Social Media Marketing: How Much Are Influentials Worth? *

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Abstract

This paper studies the competition between firms for social media influencers. Firms spend effort to convince influencers in a network to recommend their product. The results show that the value of an influencer depends on the structure of the influence networks. Influencers who exclusively cover some consumers are more valuable to firms than those who cover consumers also covered by other influencers. The total effort expended by firms is completely determined by the in-degree distribution of the network. Firm profits are highest when there are many consumers with a very low in-degree or with very high in-degree. Consumers with an intermediate level of in-degree contribute negatively to profits and high in-degree consumers increase profits especially when market competition is not intense. Prices are generally lower and social welfare is higher when consumers are covered by many influencers , however, firms are not always worse off with lower prices. An extension models the incentives firms provide to influencers and how these affect influencers' network formation behavior. Direct monetary incentives lead to dense networks and consumers covered by many influencers. Such networks are only beneficial to firms when market competition is not intense. When the market is more competitive, firms are better off providing smaller direct incentives just enough to ensure a sparse coverage of consumers by influencers.

1 Introduction

The emergence of social media is transforming the way firms approach consumers. Nielsen reports[1] that 92% of consumers trust product recommendations from people they know, vastly exceeding any form of advertising or branded communication. Recognizing the power of word-of-mouth, marketers began to reach out to influential consumers who can convince their peers more effectively than traditional advertising would.

In order to utilize the value of the vast consumer-to-consumer communication networks facilitated by social media, companies need to: i) identify influencers, and ii) convince influencers to recommend their products. There are a number of upcoming service providers offering assistance with the first task. The most notable example is Klout.com, a site that collects data about consumers from social networks to estimate their influential power. While the task of assigning individual influence scores is not easy and frequent adjustments of the methodology is necessary,[2] various startups had made good progress in this direction, some of them being able identify influencers at a very granular level.[3]

Once marketers have identified influencers the next step is to get them to recommend the marketed product. There are different approaches to convincing influencers, but all of them involve considerable effort from firms. The activities firms conduct can range from simple communications that demonstrate the value of the product through offering extra perks to directly paying influencers. For example, Cathay Pacific offers lounge access to customers (of any airline) that have a high Klout score.[4] Other companies such as online fashion retailers Bonobos and Gilt offer discounts, whereas Capital One has increased credit card rewards for customers with high influence scores. Some of these offers go to the lenght of giving away

[1] *"Global Trust in Advertising and Brand Messages",* Nielsen, April 2012

[2] *"Klout, Controversial Influence-Quantifier, Revamps Its Scores",* available at http://www.businessweek.com/articles/2012-10-18/finding-social-medias-most-influential-influencers

[3] *"Finding Social Media's Most Influential Influencers",* available at http://www.businessweek.com/articles/2012-10-18/finding-social-medias-most-influential-influencers

[4] *"Free Cathay Pacific Lounge Access at SFO via Klout.. If You Are Cool Enough,"* available at http://thepointsguy.com/2012/05/klout-offers-some-free-cathay-pacific-lounge-access-at-sfo/

products for free.[5] Thus, it is apparent that companies spend considerable effort trying to identify and win over influencers, but it is not clear what the value of each influencer is and how this depends on the influence network between consumers. This is reflect in the widespread and intense discussion among practicioners about the return on investment in social media marketing efforts.[6]

The common wisdom suggests that consumers who have many peers listening to them are valuable, but this simple prescription that only considers the reach of each influencer does not take into account the potential overlap between consumers covered by different influencers. In a competitive environment, it is crucial to understand how firms should value each influencer depending on which of their peers these influencers can have an impact on. Another commonly held view is that the more links and communication there is, the better off marketers who rely on influencers are. Again, this train of thought ignores the potential for competition and the possibility that influencers may recommend the competitor's product.

In order to rigorously study the problem of how much effort to spend on influential consumers in a competitive environment, we develop an analytical model addressing the following questions:

- How much effort should competing firms spend on each influencer?
- What is the role of the network structure and the level of competition in the value of different influentials? Are highly connected influentials always the most valuable?
- What is the effect of the influence network on prices, firm profits, consumer welfare, and social welfare?
- How should firms incentivize influentials? Should they offer direct (monetary) benefits or should they convince them in other ways?

[5] *"Gilt Gives Discounts to Match Klout Scores"*, available at http://allthingsd.com/20120305/gilt-gives-discounts-to-match-klout-scores/

[6] *"Driving Business Results With Social Media"*, - Businessweek, July 2011, availalbe at http://www.businessweek.com/managing/content/jan2011/ca20110120_489176.htm

- How will influentials change their network in response to different incentives?

The main model includes two competing firms who try to win over influencers in a network. Firms exert effort on each influencer and when they succeed the influencer recommends their product. Consumers who receive recommendations about only a single product provide a unit margin to the firm selling that product, whereas those who receive recommendations about both products provide a lower margin to both firms. The exact margin depends on the level of competition and can be as low as zero in very competitive environment.

We find that there is a symmetric equilibrium where the two firms follow the same strategies, but the effort levels for each influencer differ based on his or her position in the network. We identify the equilibrium effort in the most general fashion: for each influencer in any network. The effort level for a given influencer depends on whether consumers covered by this influencer are covered by other influencers and if yes, how many of them. Surprisingly, however, the effort levels are independent of the level of product market competition between firms. This follows from the phenomenon that firms value influencers for both offensive and defensive purposes. On one hand, winning over an influencer makes it possible to convey a message to consumers who do not otherwise receive recommendations about a product. On the other hand, winning over an influencer prevents the competing firm from having its product recommended to consumers. The combination of the two effects makes it equally important to spend effort on influencers in both a competitive and a non-competitive environment (offense pays off more in the latter, defense in the former).

Adding up the effort levels for all influencers reveals that the total effort exerted is determined by a very simple network statistic: the in-degree distribution of the influence network. Consumers with a small in-degree, those who are influenced by only a few influencers, contribute to effort levels less than those with a high in-degree. As a consequence, highly connected influencers are valuable, but only if they cover consumers who are not covered by many other influencers. Firm profits depend on the network structure in an interesting way as the profit is a U-shaped function of the in-degrees. Networks where each consumers covered by very few, and those where each consumer is covered by a large number of influencers are the most profitable, whereas networks where each consumer is covered by an intermediate number of influencers are the least profitable.

In order to study price and welfare levels, we extend the model to include firms' pricing decisions. In equilbrium, we find that firms employ mixed pricing strategies while using effort levels similar to those of the basic model to convince influencers. We find that prices depend on the network structure in an interesting fashion. In sparse networks, where consumers are typically covered by a few influencers, prices are high as firms are able to extract almost all surplus from consumers who only consider one product. At the other extreme, when consumers are under the influence of many influencers, prices go down as price competition increases. Surprisingly, these lower prices do not always hurt firms, especially when recommendations do not lead to considering a product with high probability.

Introducing pricing allows us to examine social welfare and we find an interesting pattern. In general, dense influencer networks where consumers are covered by many influencers lead to greater social welfare. The general intuition behind this finding follows from the notion that firms spend more aggressively on influencers when they have exclusive coverage. This "wasteful" spending is reduced when influencers do not have exclusive reach to certain customers. Whether most of this increased social welfare realizes as consumer surplus or firm profits depends on the particular recommendation mechanism.

Finally, we study how firms should incentivize influencers to form their networks and how consumers react to different incentives. In principal firms have two different ways making influencers recommend their product. We find that when firms use a direct, possibly monetary incentive, influencers establish more links, expanding their network. This can benefit or hurt firms depending on the level of competition. The main result is somewhat counterintuitive: when the product market is competitive firms should not offer a very high monetary benefit to influencers so as to avoid too much coverage. When the market is not that competitive, firms should encourage influencers to build their network with substantial monetary benefits.

The rest of the paper is organized as follows. In the next section, we review the relevant literature, then, in Section 3 we introduce the model. In Section 4, we derive the equilibrium of the basic model. Next, in Section 5, we examine the pricing decisions and in Section 6, we study the incentives provided by firms to influencers and the resulting endogeneous network formation behavior. Finally, in Section 7, we conclude and discuss the practical implications and limitations of our results.

2 Relevant Literature

The importance of word-of-mouth communications in marketing has been long recognized by the literature, starting with widely emplyed product diffusion model of Bass (1969). Most of the early models concerned aggregate effects, but with the new developments in network analysis and the availability of network data, academic research recognized the need to understand the role of the underlying network Goldenberg et al. (2001), Mayzlin (2002), Van den Bulte and Lilien (2003), Godes and Mayzlin (2004). First on simple, than on more complex networks the marketing literature has uncovered several important network properties and their contribution to the diffusion process Katona et al. (2011), helping to identify influencers Trusov et al. (2010). At the same time, computer scientist have also addressed the question of how to seed a network optimally to maximize viral spread Kempe et al. (2003), Stonedahl et al. (2010).

Most papers in the area focus on how a monopolist should target a few consumers in a network to optimize the diffusion of a single product or idea ignoring competition. A notable exception considering competition in a direct marketing setting is a paper by Zubcsek and Sarvary (2011) who consider competing firms advertising to a social network. They find that it is important for advertisers to take into account the existence of the underlying network, especially its density. Despite some similarities, our paper is different in several aspects. We are not interested in maximizing a single campaign, instead we analyze how firms should invest to win over influencers in a network in order to capture market share, somewhat akin to Chen et al. (2009). We also take into account that firms can exert different amount of effort engaging in a contest for each influencer. This allows us to quantify the value of each influencer which - as our findings show - depend heavily on the distribution of in-degrees.

Our paper is also related to the literature on targeting and advertising. Our model includes the ability of firms to individually target influencers, related to Chen and Iyer (2002), Chen et al. (2001). Papers such as Iyer et al. (2005) study the strategic effects of targeting and how firms can avoid intense price competition. We also share some commonalities with a recent paper on contextual advertising by Zhang and Katona (2012) who model competing firms engaging in auction for certain target segments. Finally, our treatment of pricing decisions builds on the widely adopted formulation by Varian (1980), Narasimhan (1988).

3 Model

3.1 The Consumer Influence Network

We assume that there are M consumers in the market, $N \leq M$ of which are influencers. We call consumers who can potentially affect their peers' product choices influencers[7]. We order consumers so that the first N consumers, that is $i = 1, 2, \ldots, N$, are influencers. Each influencer can potentially influence any of his or her peers, whom we call influencees.

A key feature of our model is the underlying influence network structure that determines the influencer-influencee relations. We model these connections as a very general random network. Let I_{ij} be the indicator variable showing if a product recommendation takes place ($I_{ij} = 1$ if yes, 0 otherwise) between influencer i and influencee j. When $I_{ij} = 1$, a unidirectional influence link exists between i and j. We say that influencer i covers consumer j in this case. In our analysis we use several subscripts to denote consumers, but for clarity we distinguish between influencers (denoted by i, h) and influencees (denoted by j, g) based on

[7] In theory, all consumers could be influencers (and our model allows this), but to paint a more realistic picture, we assume that only a proportion of them has enough influential power to significantly effect others' choices.

their role.

We take a very general approach to model how the random network is generated. We assume that all I_{ij} variables are independent and we use the probability

$$w_{ij} = \Pr(I_{ij} = 1) \tag{1}$$

to measure the expected strength of influence of influencer i on consumer j. We do not specifically model the nature of influence between consumers, but there are several possibilities. For example, a very simple form of influence is when one consumer makes another consumer consider a product.[8] A more complex form of influence is when consumers share positive information about a product and try to convince their peers about the product benefits. Regardless of the specific process, we assume that a higher w_{ij} corresponds to a higher likelihood of an effective correspondence taking place.

The collection of w_{ij} values yields the influence network matrix $\mathbf{W}$. We assume that $\mathbf{W}$ is common knowledge, but that the outcome of each random variable is not observed by players. One can think of $\mathbf{W}$ as the adjacency matrix of the weighted network which is obtained as the expected influence network between consumers. For example, if all $w_{ij} \in \{0, 1\}$, then the influence network is deterministic and the expected network is the same as the actual influence network. However, it is more realistic to assume that players do not observe the exact network, especially since this network is not identical to the underlying social network. Product influence rarely takes place with high probabilities on all social network links. That is, $\mathbf{W}$ represents what firms believe the influence network could be. With the emergence of social network analytic technologies and services these beliefs become more and more accurate. Companies like Klout provide measures about consumer influence levels, and although some of these measures are still rudimentary, there is rapid development is this area.

An important quantity in our analysis will be the total number of influencers that cover a particular influencee. We also need to count this quantity separately for consumers covered

[8]We use this specification in Section 5 when modeling pricing strategies.

by a particular influencer, thus, we introduce

$$\varphi_{ki} = \sum_{j=1}^{M} \Pr\left(I_{ij} = 1, \sum_{h=1}^{N} I_{hj} = k\right) \text{ and } \varphi_k = \sum_{i=1}^{N} k\varphi_{ki} \tag{2}$$

That is, φ_{ki} counts the expected number of consumers covered by i, who are influenced by exactly k influencers (including i), whereas φ_k measures the expected frequency of consumers covered by k influencers. φ_k can also be interpreted as the expected in-degree distribution of the influence network, whereas φ_{ki} is the expected degree distribution among consumers covered by influencer i. Since our setup is very general in terms of the network structure, we derive the solution for any type of degree distribution. However, in order to ensure an equilibrium in pure strategies, we assume that φ_{1i} is not too small for any influencer.[9]

3.2 Firms and Influencers

We assume that there are 2 firms ($f = 1, 2$) selling their products to the M consumers. Firms take advantage of social media and target influencers to increase the revenue they can expect from consumers. Consumers thus influence their peers to buy the product of one of these (or both) firms. Let R_f denote the set of consumers who receive recommendations only about the product of firm f, whereas R_{12} the set of those who receive recommendations about both products. In our baseline model, we assume a very simple revenue structure for firms who sell their products at zero variable cost. Each consumer $j \in R_f$ provides a revenue of 1 to firm f, whereas a consumer $j \in R_{12}$ provides a revenue of $q \geq 0$ to both firms. In essence, q measures the intensity of competition between the two firms. When $q = 0$, the two firms are competing heavily and cannot extract any profits from consumers that strongly consider both products. As q increases to $1/2$ the competition becomes less intense and while the products are still substitutes, firms can extract the same amount of profit from these consumers as from those only considering one product. As q increases further, the two products eventually become complements.

[9]Formally, we need $\varphi_{1i} \geq (1/4) \sum_{k=1}^{\infty} \varphi_{ki}$. This is a rather technical assumption which does not put too much restriction on network structure, especially if the w_{ij} probabilities are relatively small.

Although we do not directly model the product market competition in this basic model, there are several micromodels that would produce results corresponding to our assumption. For example, if we assume that personalized pricing is possible, Bertrand competition would lead to $q = 0$ when consumers in R_{12} compare the two products. Higher q values could be obtained using a bargaining or differentiation models. Furthermore, even if personalized pricing is not possible, and all consumers pay the same price, one can produce results consistent with this assumption. In the next Section 5, we present such an extended version of the model, where firms make pricing decisions and show that it can correspond to our basic model with any $0 \leq q < 1$.

We assume that the N influencers have a limited bandwith to talk to their peers and they only recommend one product. This is consistent with survey results[10] showing that influencers recommend a limited number of products, most of them less than ten per year. Since this already low number of recommendation is split over different product categories, it is plausible to assume that most influencers recommend only one of two competing products. To become the one recommended brand, firms invest resources to convince influencers to recommend their product. For each influencer, let e_{fi} denote the effort spent by firm f on influencer i. We assume that the probability that an influencer recommends a firm is proportional to the effort spent by firms. Formally,

$$\Pr(i \text{ recommends firm } f) = P_{fi}(e_{1i}, e_{2i}) = \frac{e_{fi}^{1/r}}{e_{1i}^{1/r} + e_{2i}^{1/r}}. \tag{3}$$

This is called a Tullock success-function (Tullock 1980), which is often used to model contests and all-pay auction in the economics literature. The parameter r measures the softness of competition for winning over an influencer. The lower the r, the more firm efforts affect the probability that the firm is picked by an influencer.[11] We assume $r > 1/2$ to ensure a

[10] *"Three Surprising Findings About Brand Advocates"*, Zuberance Whitepaper, February 2012, available at http://zuberance.com/brandadvocateresearch/

[11] Another way to interpret r is as the exponent of the cost of effort. If we assume a strictly proportional success function $\frac{e_{fi}}{e_{1i}+e_{2i}}$, then the above formulation is equivalent to assuming that the cost of effort is $c(e) = e^r$.

pure-strategy equilibrium. As a result of firm efforts, influencers recommend the appropriate products and consumers receive recommendations about either one or both products. Thus, the payoff of firm f becomes

$$\pi_f = (|R_f| + q|R_{12}|) - \sum_{i=1}^{N} e_{fi}. \tag{4}$$

Note that we do not explicitly model how firms convince influencers to recommend their products, instead we take a general approach and only focus on how much firms spend. In Section 6, we explore an interesting extension, where firms can provide different types of incentives to influencers.

4 Equilibrium Analysis

Our first goal is to determine the equilibrium effort that firms invest in each influencer. Since the game is symmetric, we will focus on the symmetric equilibrium which, as we will show, is typically the unique pure-strategy equilibrium. In order to determine the optimal effort level for influencer i, let $N_i = \{j : I_{ij} = 1\}$ denote the set of consumers that i has influence over. Let us examine the decision of firm 1 for influencer i. Assume that all of firm 2's effort levels are fixed and that firm 1's effort levels for influencers other than i are also fixed. That is, P_{1h} is also fixed for all $h \neq i$ influencers. Since we are focusing on a symmetric equilibrium, we can assume that $P_{1h} = 1/2$ for all $h \neq i$. Then the payoff of firm 1 can be written as

$$\mathbf{E}\,\pi_1 = \sum_{g=1}^{M} [\Pr(g \in R_1) + q\Pr(g \in R_{12})] - e_{1i} - \sum_{h \neq i} e_{1h}. \tag{5}$$

Thus, we write the revenue part of the expected profit as a sum where, for each consumer g, we calculate the expected revenue of that consumer. The expected revenue is the probability that the consumer only gets recommendations for product 1 plus q times the probability that s/he gets recommendations for both products. Note that these probabilities do not change for consumers that are not covered by N_i, therefore we restrict our attention to the part of $\mathbf{E}\,\pi_1$ that varies with e_{1i}:

$$\sum_{g\in N_i}[\Pr(g\in R_1)+q\Pr(g\in R_{12})]-e_{1i} \tag{6}$$

These probabilities further depend on how many other influencers cover g. Let k denote the total number of influencers (including i) that have g in their influence set. For example if $k=1$, the only way to access consumer $g\in N_i$ is through influencer i. Thus,

$$\Pr(g\in R_1)+q\Pr(g\in R_{12})=P_{1i}(e_{1i},e_{2i})=\frac{e_{1i}^{1/r}}{e_{1i}^{1/r}+e_{2i}^{1/r}}, \tag{7}$$

and this consumer will yield a revenue of 1 iff influencer i recommends firm 1 (the probability of which is P_{1i}). When $k>1$,

$$\Pr(g\in R_1)+q\Pr(g\in R_{12})=\frac{1}{2^{k-1}}P_{1i}+q\left(\frac{2^{k-1}-1}{2^{k-1}}P_{1i}+\frac{2^{k-1}-1}{2^{k-1}}(1-P_{1i})\right)=$$
$$\frac{1}{2^{k-1}}\cdot\frac{e_{1i}^{1/r}}{e_{1i}^{1/r}+e_{2i}^{1/r}}+q\frac{2^{k-1}-1}{2^{k-1}}. \tag{8}$$

Note that the coefficient of q does not depend on e_{1i}, therefore, we can write the profit as

$$\mathbf{E}\,\pi_1=\frac{e_{1i}^{1/r}}{e_{1i}^{1/r}+e_{2i}^{1/r}}\sum_{k=1}^{\infty}\frac{\varphi_{ki}}{2^{k-1}}-e_{1i}+C, \tag{9}$$

where C is a constant that does not depend on e_{1i}. The above formula tells us that the value of influencer i for firm 1 depends only on how many other influencers cover i's influencees. For a particular g influencee of i that is covered by exactly k influencers the value is exactly $\frac{1}{2^{k-1}}$. To derive the equilibrium effort levels, we simply take the first order condition and obtain the symmetric equilibrium. The following proposition summarizes the main results.

Proposition 1

1. The symmetric equilibrium effort level for firm f and influencer i is

$$e_{fi} = \frac{1}{4r} \sum_{k=1}^{\infty} \frac{\varphi_{ki}}{2^{k-1}}.$$

2. The total effort levels and the profits are

$$\sum_{i=1}^{M} e_{fi} = \frac{1}{4r} \sum_{k=1}^{\infty} \varphi_k \frac{k}{2^{k-1}}, \quad \pi_f = \sum_{k=1}^{\infty} \varphi_k \left(\frac{2 - k/r}{2^{k+1}} + q \frac{2^{k-1} - 1}{2^{k-1}} \right).$$

3. If q is not too large, the above equilibrium is unique.

The results highlight various interesting features of the equilibrium. The first part provides a very important quantity: the amount a firm invests to convince an individual influencer to recommend its product. The formula shows a nice pattern as the equilibrium effort level is a sum with components corresponding to each consumer that the influencer has potential influence on. Each individual that is only influenced by i contributes $\frac{1}{4r}$ to the equilibrium effort level for i. This is consistent with the literature on contests, as this is the exact effort level that one of two competitors incurs for a prize worth 1 unit. That is, each influencee that is only covered by influencer i contributes 1 unit to i's worth. However, each influencee that is covered by exactly k influencers (i plus $k-1$ others) contributes exactly $\frac{1}{2^{k-1}}$ to i's worth. Surprisingly, this value and the entire effort level does not depend on q. The intuition for this unexpected result is based on the following: when an influencee is covered by many influencers, it is very likely that this person receives recommendations about both products. Therefore, firms are mostly indifferent whether they win over influencer i or not, thus influencees covered by many influencers do not contribute too much to the value of the influencer. In the unlikely case when a person covered by many influencers other than i gets recommendations about only one firm, s/he is equally likely to get the recommendation about firm 1 or firm 2. In the former case, the value of winning over influencer i is $1-q$, in the other it is q, which sum up to 1, regardless of q.

Another way to understand this result is by thinking about two different purposes of winning over an influencer. One is the offensive reason, to reach consumers who only receive recommendations about the other product. The offensive benefit from such a consumer is q. The other purpose is to prevent comepetitors from reaching consumers who only receive recommendations about one firm's product. The defensive value is $1-q$. In the symmetric equilbrium the two purposes are equally likely to be at play, thus, q is canceled out.

The second part of the proposition demonstrates that the total effort spent by one firm also does not depend on q. Interestingly, the network structure determines the total effort and profits in a very simple way. If one knows the expected number of individuals covered by exactly k influencers, the total effort and profits are simply a weighted sum of these φ_k coefficients. Figure 1 shows the weights. When $q=0$, the weight of φ_k is $\frac{2-kr}{2^{k+1}}$ which is a U-shaped function. That is, in a very competitive market only individuals who are only covered by exactly one influencer contribute positively to profit. The number of individuals covered by exactly two influencers does not change profits, whereas those covered by more than two influencers decrease profits. However, those covered by a large number of influencers have only a small negative effect as firms realize that competition is tough and decrease their efforts for highly covered consumers.

When $q>0$ the profit contribution of highly covered individuals changes in an interesting way. As $k\to\infty$, the contribution of a consumer covered by k influencer converges to q. This results from the lowered effort spent on influencers covering these consumers. But despite the reduced incentives to invest, these consumers contribute to profits positively. As they most likely receive recommendations about both firms, they yield a profit of q to each firm.

It is worthwhile to examine how the two parameters measuring competition affect the results. Recall that q measures the softness of the product market competition, whereas r measures the softness of the competition for winning over influencers. As one would expect, profits are increasing in both r and q. As competition becomes softer in either area, firms are better off. However, the value of consumers with different local network structures changes in

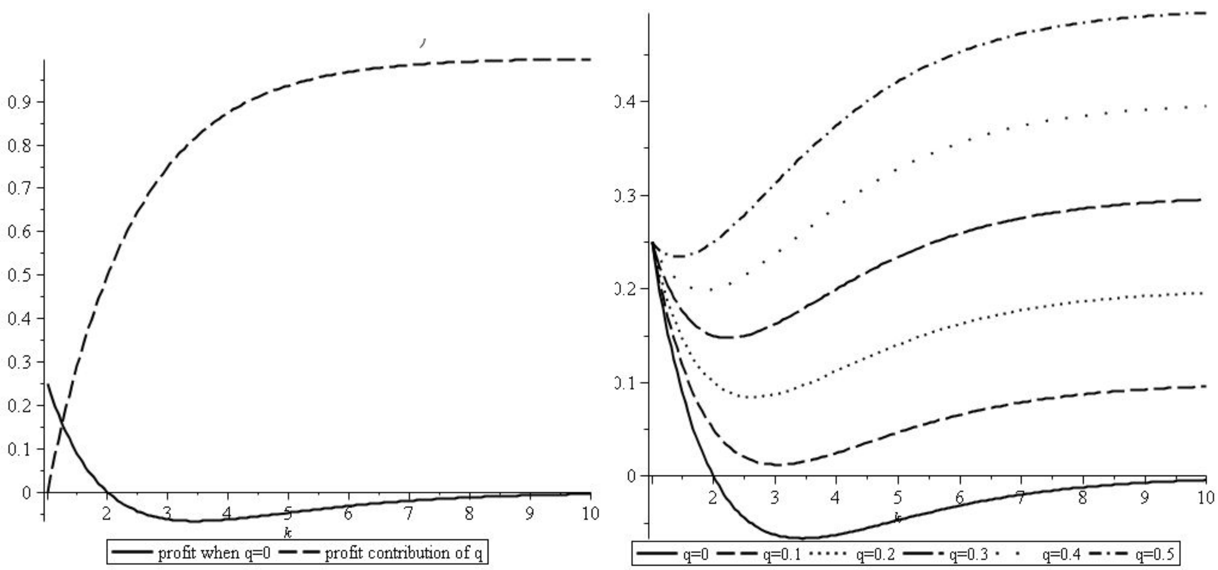

Figure 1: The contribution of different in-degrees to profits. On the left, the solid curve represents $q = 0$ while the dashed curve shows the component multiplied by q. On the right, the profit is depicted for different values of q.

an interesting way with these parameters as the following corollary summarizes.

Corollary 1

1. *There exists a $\overline{k}$ such that consumers covered by $k > \overline{k}$ influencers contribute to profits more than those covered by* 1 *influencer if and only if* $q > \frac{1}{2} - \frac{1}{4r}$.

2. *Consumers covered by exactly* $\lfloor 2(1 + (1 - 2q)r) \rfloor$ *influencers contribute the least to firm profits. Their contribution is negative when q and r are low.*

Firstly, since the profit contribution of a consumer covered by k influencers is a U-shaped function of k, it is not clear which consumers are worth the most. As we show above, the consumers who are worth more are either the ones covered by only one influencers or the ones covered by all of them. If $q > \frac{1}{2} - \frac{1}{4r}$, that is, if competition is generally soft (q and/or r are high), then consumers covered by all influencers are the most valuable in their contribution to profits. When competition either in the product market or for influencers is tough, then we get the opposite results and influencers that have a large exclusive coverage are worth the most.

Secondly, consumers who are covered by an intermediate range of influencers are the least valuable to firms as firms have less incentive to win over influencers covering such consumers. When competition is tough, firms will want to avoid these consumers and the influencers covering them as they decrease profits.

5 Price Competition

In our basic model we took a very generic approach to model the nature of influence and competition. We assumed that firms made a fixed amount of profit per consumer which depended on whether consumers received recommendations about only one or both products. Here, we extend the model to incorporate the pricing decisions. In order to do so, we model the process of product recommendations between consumers in more detail. If a consumer receives a product recommendation from an influencer, that product enters his/her consideration set with probability γ independently from recommendations about competing products. Multiple recommendations about a single product do not have a cumulative effect.[12]

Each consumer has a reservation price normalized to 1 for each product. If there is only one product in the consumer's consideration set, s/he purchases that product as long as the price does not exceed his or her reservation price. When there are multiple products in his or her consideration set, the consumer chooses the lowest priced product.

We assume that firms set their p_f prices simultaneously at the same time when determining their e_{fi} effort levels. That is, all decisions by all firms are made at the same time. We also assume that firms do not have the ability to charge a different price to different customers.[13]

Again, we focus on symmetric equilibria. It is clear that an equilibrium with pure strategies in prices does not exist, since there is always a positive mass of consumers in expectation who are aware of at least two products. The firms selling these products thus have an incentive to slightly lower their price.

We find the symmetric equilibrium with pure strategies in setting the effort levels, but with mixed strategies in the pricing decision, similarly to Varian (1980) and Narasimhan (1988). The following proposition summarizes the equilibrium results on prices and effort levels:

Proposition 2

1. *In the unique symmetric equilibrium firms set random prices with support*

$$\left[1-\gamma\left(1-\frac{\sum_{k=1}^{\infty}\frac{\varphi_k}{2^k}}{\sum_{k=1}^{\infty}\frac{\varphi_k(2^k-1)}{2^k}}\right),1\right] \text{ and p.d.f. } g(p)=\frac{1}{p^2}\left(\frac{1}{\gamma}-1+\frac{\sum_{k=1}^{\infty}\frac{\varphi_k}{2^k}}{\sum_{k=1}^{\infty}\frac{\varphi_k(2^k-2)}{2^k}}\right).$$

[12] That is, we assume that recommendations have extremely decreasing returns. The first recommendation about a product is successful in changing the consideration set with probability γ and subsequent recommendations about the same product are ignored. One can imagine a setting where multiple recommendations have decreasing but positive returns. Such a model would lead to similar intuition.

[13] The results are essentially the same if firms can set a different price for each customer.

2. The equilibrium effort level for firm f and influencer i is

$$e_{fi} = \frac{\gamma}{4r} \sum_{k=1}^{\infty} \frac{\varphi_{ki}}{2^{k-1}}.$$

3. The total effort levels and the profits are

$$\sum_{i=1}^{M} e_{fi} = \frac{\gamma}{4r} \sum_{k=1}^{\infty} \varphi_k \frac{k}{2^{k-1}}, \quad \pi_f = \gamma \sum_{k=1}^{\infty} \varphi_k \left(\frac{2 - k/r}{2^{k+1}} + (1-\gamma)\frac{2^{k-1}-1}{2^{k-1}} \right).$$

The results demonstrate how firms set prices in conjunction with their effort levels. The first part describes the equilibrium price distribution showing that firms set random prices up to 1 starting at an intermediate value. The size of the interval and thus the overall level of prices depends on the network structure in an interesting way. When most consumers are only covered by a few influencers, prices are generally high as firms are able to extract most surplus from consumers. Since the influencers have limited bandwidth and they only recommend one product few consumers consider multiple products in this case leading to little price competition and high prices. When most consumers are covered by many influencers, prices decrease as many consumers consider both products due to multiple recommendations. This leads to increased price competition and lower prices. Interestingly lower prices do not always hurt firms. When γ is sufficiently low and consumers are not likely to consider a recommended product, increased coverage by influencers lowers prices, but increases profits at the same time. The intuition follows from a combination of two strategic effects. On one hand increased coverage results in tougher price competition and lower prices. On the other hand high coverage leads to carefully investments in the effort to convince influentials, leading to less wasteful spending.

In terms of effort levels, the results are similar to those of Proposition 1. For example, when $\gamma = 1$, we get the same results as if we set $q = 0$ in the basic model. However, despite the mathematical similarities, the results are conceptually different. It is not the case that firms make 0 profit on consumers that consider both products and 1 on those who consider only 1 product. Instead, the mixed pricing strategies yield an equilibrium where firms extract surplus from all consumers, but naturally more from those who consider only one product. Similarly to Varian (1980), Narasimhan (1988), the total surplus extracted by a firm equals the profit that a firm would get if all consumer only considering its product would pay their reservation price. As γ decreases from 1, we uncover two effects. Naturally, consumers who receive a recommendation only about one firm will be less likely to consider the product, that is the overall value of the market decreases. However, consumers who receive recommendations about both products will be less likely to consider both products, leading to a higher effective $q = 1 - \gamma$. Thus, a less effective recommendation decreases the value, but also reduces competition at the same time. Surprisingly, this leads to profits that are decreasing in γ when γ is close to 1 and less than half of the consumers are influenced by only one influencer ($\varphi_1 < M/2$).

It is useful to understand the social welfare implications in order to contribute to the public policy debate on the value of advertising in social media.

Corollary 2 *The social welfare is*

$$\gamma \sum_{k=1}^{\infty} \varphi_k \left(2 - \gamma - \frac{\frac{k}{2r} + 1 - \gamma}{2^{k-1}} \right)$$

which is increasing in r and, increases as consumers are covered by more influencers.

The results paint an interesting picture. Figure 2 shows social welfare, consumer surplus, and firm profits. Although firms profits do not always increase as consumers are covered by more influencers, social welfare does. The main reason lies in the intensity of firm efforts spent on convincing influencers. If we think of this as wasteful spending,[14] it becomes clear why more coverage leads to higher social welfare: when consumers are covered by many influencers, firms are much less aggressive in convincing influencers given the reduced value they represent.

[14]One could account for some of this amount as part of social welfare if influencers are directly paid or third parties profit from it.

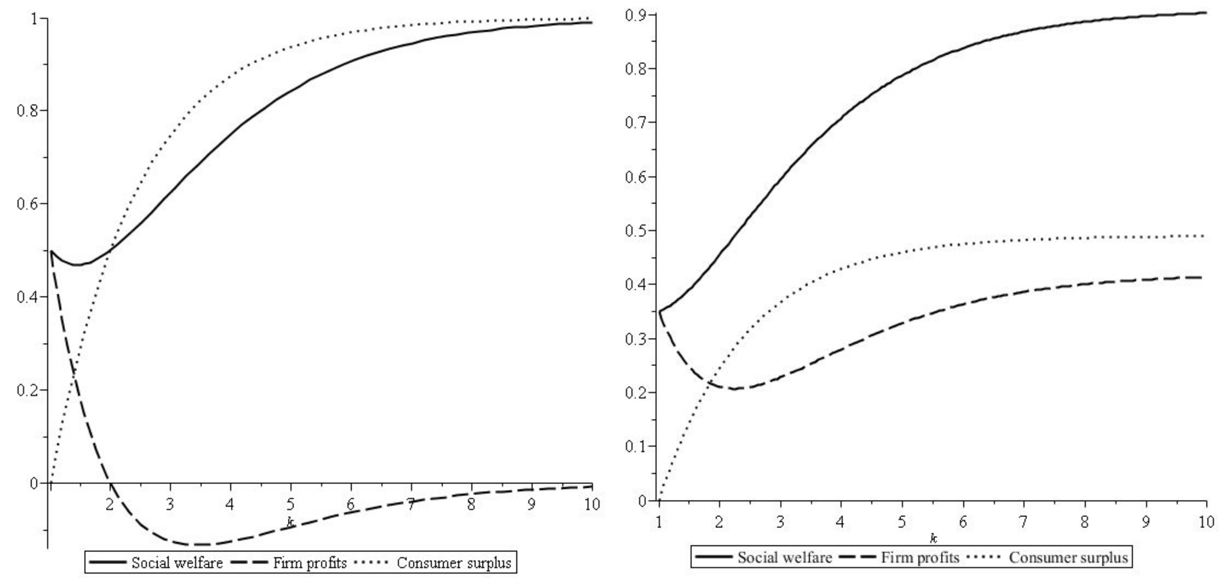

Figure 2: The contribution of consumers with different in-degrees to social welfare, firm profits, and consumer welfare. On the left, $r = 1, \gamma = 1$. On the right $r = 1, \gamma = 0.7$

6 Endogeneous Network Formation and Influencer Incentives

Throughout the paper, we have assumed that the influence network was exogenously given. However, there is increasing pressure on consumers to become influential among their peers in order to get perks and benefits from firms. Klout - the influence score tracking service - even created a Klout Perks API that makes it easier for firms to offer perks to their highly influential customers.[15] In this section, we examine how endogenizing the influence network formation affects our results. Furthermore, we study how firms can impact the network formation behavior by providing direct incentives to consumers.

First, we describe the network formation process. As before, we assume that there are N

[15] *"Your Influence (and Klout Score) is Worth Money"* - available at http://www.stateofsearch.com/your-influence-and-klout-score-is-worth-money/

influencers out of M consumers in the market. Consumers are heterogeneous in terms of how hard they are to influence and each consumer is described by a c_j parameter. Whenever an influencer i wants to establish an influence link of w_{ij} strength (probability), the influencer has to expend $c_j w_{ij}$ effort. In other words, c_j measures the cost of being able the fully influence consumer j for any single influencer. For the sake of tractability, we assume that the cost parameter is uniformly[16] distributed between 0 and $\overline{c} < 1/2r$. Formally, let $c_i = \frac{\overline{c}}{M} \cdot i$.

In order for influencers to build their networks and attempt to influence their peers, they have to receive some benefit. We assume that this benefit comes directly from firms. When firm f exerts e_{fi} effort to convince influencer i, some of this effort directly benefits influencer i as a monetary transfer. The parameter α_f measures the percentage of the effort e_{fi} that directly goes to the influencer. That is, in our model influencer i receives a payment of $\alpha_f e_{fi}$ from firm f. In reality this payment does not have to be cash, instead it can be in the form of a rebate, points for future purchases, or simply a price reduction. We do not model the exact nature of the payment, our assumption only specifies that $\alpha_f e_{fi}$ directly increases i's utility, whereas the remaining $(1 - \alpha_f)e_{fi}$ does not change i's utility, only his or her probability to recommend firm f.

We assume that the above decisions take place in the following sequence. First, firms simultaneously announce and commit to α_f, the percentage of effort that directly benefits influencers. Second, influencers simultaneously build the network by selecting each consumer they want to influence and the strength of the relationship. Third, firms simultaneously determine their effort levels, exactly as in Section 3. As before, we are looking for equilibria that are symmetric with respect to the firm strategies and we also require subgame-perfection. We first determine the possible equilibrium networks for fixed α_f.

Lemma 1 *For fixed α_1 and α_2 values, a W random influence network is an equilibrium outcome if and only if all $w_{ij} \in \{0, 1\}$ and for any consumer j, the equation $\sum_{i=1}^{N} w_{ij} = \left\lceil \log_2 \left(\frac{\alpha_1 + \alpha_2}{r c_j} \right) - 2 \right\rceil$ holds.*

[16] The results are similar for various other distributions.

The lemma shows us that equilibrium networks only have deterministic links,[17] but there are multiple such equilibria. The only requirement for an influence network to be an equilibrium outcome is a restriction on how many influencers can cover a given consumer. In equilibrium, any consumer j whose cost parameter falls in the interval $\frac{\alpha_1+\alpha_2}{2^{k+2}r} \leq c_j < \frac{\alpha_1+\alpha_2}{2^{k+1}r}$ has to be in the influence set of exactly k influencers. Although the equilibrium is not unique as there is no restriction on which influencer covers which particular consumer, the above condition determines the φ_k values. This allows us to solve for the equilibrium α_f levels.

Proposition 3 *For low values of q, we have $\alpha_1 = \alpha_2 = 2r\bar{c} < 1$ in the symmetric equilibrium. For high values of q, we have $\alpha_1 = \alpha_2 = 1$ in equilibrium.*

The result shows a surprising pattern. When the market is not very competitive, firms spend all their efforts on directly increasing the utility of influencers. The reason is that a high percentage of direct payoff incentivizes influencers to build a dense network that benefits firms given the low level of competition (high q). On the other hand, when competition is intense, firms do not want to provide such strong incentives as too dense coverage by influencers would hurt their profits. But firms do not want to provide too little incentive as that would leave many consumers untouched by influencers. Therefore, firms set the percentage of direct benefit just high enough so that influencers cover the entire market. In the resulting network, half of the consumers will be covered by only one influencer, in contrast to the case of $\alpha_1 = \alpha_2$, where most consumers are covered by many influencers. It also intersting to examine how the α_f levels are affected by r. When r is low and the competition for influencers is intense, firms are more likely to offer the entire amount they spend on influencer as a direct benefit to them. When r is high, and the competition for influencers is softer firms prefer to have a sparser network, hence they offer a low percentage of direct benefit. However, this low percentage is increasing in r as the low overall effort levels in a soft competition are not enough to convince influencers to expand their coverage.

[17]This is an implication of the linear cost of influencing ($c_j w_{ij}$), which is a technical assumption to make the analysis simpler. A convex cost function would lead to more realistic results similar in nature, but more complicated to describe.

7 Conclusion

In this paper, we have explored the value of influencers in a social network to competing firms. We determined how much each influencer is worth to firms and how much effort firms are willing to spend on influencers in order to win them over. The results show that the value of the influence network is determined by its in-degree distribution. Depending on the level of competition, consumers covered by very few or many influencers contribute most to profits, whereas consumers covered by an intermediate number of influencers often reduce firm profits.

Our results have important implications for social media marketers considering the avenue of influencer marketing. First and foremost, the value of an influencer cannot be described only by the connectivity or reach of the person. One might have numerous connections, but these are often worthless unless they provide exclusive access to some consumers. As a result, seemingly isolated influencers with only a small set of connection can be valuable if their friends cannot be reached otherwise. Any type of influence score that does not take these network properties into account can be misleading. The good news is that a fairly simple statistic, such as the in-degree distribution of consumers covered by an influencer can be very informative.

Second, competition plays a very important role in determining the role of influencers. Even if a marketer is a monopolist in its own product category, firms have to compete across categories for winning over influencers as most of them recommend only a few brands. An important implication of competition is that one cannot forget about both the offensive and defensive roles of influencer marketing. The reason to win over an influencer is not always to gain additional reach to customers, but it could very well be to prevent a competitor from also reaching one's own customers.

Third, our analysis of the incentives given to influencers sends a clear, but unexpected message for social media marketers, especially if they plan to use these tools on the long term. If a firm provides too much direct, potentially cash, benefits to influencers over time, some of them may alter their behavior in order to take advantage of these incentives. This can lead to increased acitivity in becoming more an more influential, which is only good for firms up to a certain point. If the influence coverage expands too much, most consumers will be covered by multiple influencers. This, in turn, will reduce firm profits in a competitive market.

Despite the fairly general approach and solutions, our work has a number of limitations. First, we assume that there are only two firms in the market and that they are symmetric. In the Online Appendix, we explore the case of multiple firms and find that the results are similar, but tedious to describe unless $q = 0$. The case of asymmetric firms would be worthwhile to explore as the difference between the offensive and defensive benefits would be accentuated. Second, for each influencer we have a simple contest success function that ensures that the influencer will recommend one or the other firm. In reality, there might be an outside option and influencers would not recommend either firm if the efforts are not very high. This would give an incentive for firms to beat not only their competitor, but also the outside option. We explore this version of the model to some extent in the Online Appendix. Finally, we take a very simplistic approach with respect to what influencers do and whether consumers believe them. The credibility of influencers in an environment where firms heavily invest in trying to convince them is an exciting topic for future research.

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Appendix

Proof of Proposition 1: We first determine the individual effort levels that each firm puts out for a given influencer i assuming that the equilibrium is symmetric. Equations (5)-(9) express the profit as a function of e_{fi}, keeping all other effort levels fixed. We can do the same exercise for both firms and see that

$$\mathbf{E}\,\pi_1 = \frac{e_{1i}^{1/r}}{e_{1i}^{1/r} + e_{2i}^{1/r}} v_i - e_{1i} + C_1, \quad \mathbf{E}\,\pi_2 = \frac{e_{2i}^{1/r}}{e_{1i}^{1/r} + e_{2i}^{1/r}} v_i - e_{2i} + C_2, \tag{10}$$

which is a symmetric Tullock-contest with value

$$v_i = \sum_{k=1}^{\infty} \frac{\varphi_{ki}}{2^{k-1}}. \tag{11}$$

Differentiating player 1's profit with respect to e_{1i}, we get

$$\frac{\partial\,\mathbf{E}\,\pi_1}{\partial e_{1i}} = \frac{e_{1i}^{1/r-1} e_{2i}^{1/r}}{r\left(e_{1i}^{1/r} + e_{2i}^{1/r}\right)^2} v_i - 1 \tag{12}$$

The expected profit is concave in e_{1i}, hence the F.O.C give the unique maximum. Setting $e_{1i} = e_{2i}$, the F.O.C becomes

$$\frac{v_i}{4re_{1i}} = 1 \tag{13}$$

yielding $e_{1i} = \frac{v_i}{4r}$ which is the effort level given in the first part of the proposition. Note that $r > 1/2$ is necessary to ensure that the function in (10) have a positive derivative in 0. When $r < 1/2$, no pure strategy equilibrium exist.

We have determined that the effort levels given above are optimal if the effort levels for all other influentials are fixed. This sufficient to show that the effort levels in equilibrium must be these, but we need to also make sure that firms do not have an incentive to deviate by changing multiple effort levels. We will discuss this at the end of the proof together with part 3.

For part 2, we need to simply sum the effort levels across all influencers. Recalling that $\sum_{i=1}^{N} \varphi_{ki} = k\varphi_k$, we easily get the first equation for $\sum_{i=1}^{N} e_{fi}$. In order to determine profits, we need to calculate the revenues. From a consumer covered by k influencers there are two possibilities firm 1 could earn money. With probability $\frac{1}{2^k}$, the consumer is in R_1 and the revenue for firm 1 is 1. With probability $1 - \frac{1}{2^{k-1}}$ the consumer is in R_{12} yielding a revenue of q. Therefore the total revenue for firm 1 is $\frac{1}{2^k} + q\left(1 - \frac{1}{2^{k-1}}\right)$. Subtracting the investment in effort, we obtain the profit as given in the proposition.

For part 3, let us denote by $\mathcal{G}$ the game we are analyzing that has two players. Let $\mathcal{G}'$ denote a modified game with $2N$ players, with two players for each influencer. The games $\mathcal{G}$ and $\mathcal{G}'$ are identical, except that in $\mathcal{G}'$ we split each of the two players in $\mathcal{G}$ into N different players that set the N effort levels separately. Each of the N players assigned to player 1 of $\mathcal{G}$ gets the entire payoff. It is clear that the N players in $\mathcal{G}'$ face the same maximization problem that player 1 in $\mathcal{G}$ for each variable. Therefore, any equilibrium of $\mathcal{G}$ also has to be an equilibrium of $\mathcal{G}'$, but not vice versa. However, if we show that $\mathcal{G}'$ has a unique equilibrium in pure strategies $\mathcal{G}$ either has no equilibrium or a unique equilibrium. In order to show that $\mathcal{G}'$ has a unique equilibrium in pure strategies, let us write the profit function of player 1 as

$$\mathbf{E}\,\pi_1 = \sum_{S\subseteq\{1,\ldots,N\}} A_S \prod_{i\in S} P_1 i - \sum_{h=1}^{N} e_{1h}. \tag{14}$$

This formulation simply says that the revenue is a multinomial of $P_{11}, \ldots, P_{1N}$ with coefficients A_S, where $P_{1i} = \frac{e_{2i}^{1/r}}{e_{1i}^{1/r} + e_{2i}^{1/r}}$. Simple calculation shows that that when q is small enough all coefficients are positive. All the players' profit functions are concave in this case, yielding a unique equilibrium for $\mathcal{G}'$.

In order to complete the proof, we need to make sure that the equilibrium candidate generated from $\mathcal{G}'$ is also an equilibrium of $\mathcal{G}$. Thus, we need to check if for any set of S influencers that a firm (e.g. firm 1) does have an incentive to change effort levels e_{1j} jointly for $j \in S$. Given the functional form of the profit, the only way this might be profitable is if firm 1 sets $e_{1j} = 0 \forall j \in S$. Let us check whether this is profitable for $S = \{1, \ldots, N\}$. By setting all effort levels to 0, the profit would also fall to 0, that is, we only need to check that profits in the equilibrium candidate are positive. Since profits are increasing in q, the worst case scenario is $q = 0$. Examining the profit function we see that if φ_1 is high enough relative to other φ_k's, the profit is positive. Doing the same exercise for all subsets S shows that the same requirement on φ_{1i} for every i ensures that deviation is not profitable. □

Proof of Corollary 1: For part 1, Proposition 1 shows that the profit contribution of a consumer covered by k influencers is

$$a(k) = \frac{2 - k/r}{2^{k+1}} + q\frac{2^{k-1} - 1}{2^{k-1}}. \tag{15}$$

This takes the value of $\frac{1}{2} - \frac{1}{4r}$ for $k = 1$ and increasingly converges to, but never reaches q as $k \to \infty$. If $q \leq \frac{1}{2} - \frac{1}{4r}$, the maximum contribution is always for $k = 1$. If $q > \frac{1}{2} - \frac{1}{4r}$, let $\overline{k}$ denote the largest solution of

$$\frac{2 - k/r}{2^{k+1}} = q\frac{2^{k-1} - 1}{2^{k-1}}. \tag{16}$$

For $k > \overline{k}$ we get that the RHS exceeds the LHS, completing the proof of part 1.

For part 2, further examining the $a(k)$ functions shows that it first decreases, has a unique minimum then increases. To determine the minimum, we solve $a(k) = a(k + 1)$ yielding $k = 1 + 2r(1 - 2q)$. This is not necessarily an integer, but we know that the integer that minimizes $a(k)$ is between k and $k + 1$, yielding the stated formula.

□

Proof of Proposition 2: We first determine the pricing strategies. In a symmetric equilibrium, all influencers recommend both firms with the same probability, which allows us to determine the number of consumer who consider both products (denote by β) or only one product α. As we have already calculated in the proof of Proposition 1,

$$|R_1| = |R_2| = \sum_{k=1}^{\infty} \frac{\varphi_k}{2^k}, \quad |R_{12}| = \sum_{k=1}^{\infty} \varphi_k \frac{2^{k-1} - 1}{2^{k-1}}$$

Since consumers only consider a product with γ probability if they receive a recommendation, we obtain $\beta = \gamma^2 |R_{12}|$, and $\alpha = \gamma(|R_1| + (1-\gamma)|R_{12}|$. Given these quantities, we can determine what the equilibrium pricing strategies are: Following the lines of deriving the mixed strategy equilibrium in Varian (1980), we get that players mix prices in the $\left[\frac{\alpha}{\alpha+\beta}, 1\right]$ interval with p.d.f. $g(p) = \frac{\alpha}{\beta p^2}$. Plugging in the α, β values we obtain the pricing strategies described in the proposition.

When firms follow the pricing equilibrium described above in a symmetric equilibrium, their profits are α. For example, firm 1 makes a profit of γ for each consumer in R_1 and $\gamma(1 - \gamma)$ in R_{12}. Therefore, deriving the effort levels follows the same lines as the proof of Proposition 1, after we account for how much firms value consumers in R_1 and in R_2. The decision in effort levels will be very similar to the basic model, but we have to set $q = 1 - \gamma$ and renormalize the value of a consumer getting only one recommendation to γ. These will yield the stated formulas. □

Proof of Corollary 2: We define social welfare as the sum of consumer welfare and firm profits. That is, we consider the spending on effort levels wasteful spending. This is somewhat restrictive as the effort spending could go directly to influencers or to third party services, but even then some it is wasteful. To calculate social welfare in this way we simply take the expected total surplus per consumers that is to be divided between firms and subtract the spending effort. The that the consumer covered by k influencers will consider at least one product is

$$\gamma \frac{1}{2^{k-1}} + (1 - (1-\gamma)^2) \frac{2^{k-1} - 1}{2^{k-1}} = \gamma \left(2 - \gamma - \frac{1-\gamma}{2^{k-1}} \right) \tag{17}$$

Since the reservation price per consumer is 1, this probability is the expected surplus for a consumer that is to be divided between the firms and the consumer. Subtracting the effort spending gives the social welfare stated in the corollary. □

PROOF OF LEMMA 1: For a given influence network we the effort levels for each influencer from Proposition 1 as $e_{fi} = \frac{1}{4r} \sum_{k=1}^{\infty} \frac{\varphi_{ki}}{2^{k-1}}$. When α_1 and α_2 is fixed, influencer i receives a direct payment of $(\alpha_1 + \alpha_2)e_{fi}$. That is, each consumer covered by influencer i, who is covered by k influencers contributes $\frac{\alpha_1+\alpha_2}{2^{k+1}r}$ to influencer i's profit. When influencer i makes the decision on how much to influence consumer j (that is, when setting w_{ij}), he or she maximizes

$$\left(\mathbf{E} \, \frac{\alpha_1 + \alpha_2}{2^{K+1}r} - c_j \right) w_{ij},$$

where K is the random variable counting the coverage level of consumer j. Since this function is linear in w_{ij} the optimum is $w_{ij} = 0$ or 1 depending on. Therefore, in equilibrium the network is deterministic and $\mathbf{E} \, \frac{\alpha_1+\alpha_2}{2^{K+1}r} = \frac{\alpha_1+\alpha_2}{2^{k+1}r}$, where $K \equiv k$. Thus, in equilibrium $\frac{\alpha_1+\alpha_2}{2^{k+2}} \leq c_j < \frac{\alpha_1+\alpha_2}{2^{k+1}}$ has to hold for influencer j, who is covered by k influencers. Note that we do not determine how many consumers each influencer chooses to cover as there are multiple equilibria. The equilibrium requirements only determine the relationship between the cost of influencing a consumer and the number of influencers covering that consumer. □

PROOF OF PROPOSITION 3: The previous lemma shows that there is a direct link between $\alpha_1 + \alpha_2$ and the coverage levels of influencers. Examining firm profits in Proposition 1 shows that when q is high enough, firms want as high coverage as possible, therefore they set $\alpha_1 = \alpha_2$. When q is low, firms generally want lower coverage, therefore they decrease the α values. However, when reaching $\frac{\alpha_1 + alpha_2}{4r} = \bar{c}$, furhter decreases in α's do not change the relative frequency of consumers with different coverage levels. Rather the total consumer covered will decrease. Thus, at $\alpha_1 + \alpha_2 = 4r\bar{c}$ no firm has an incentive to either decrease or increase α_f. In a symmetric equilibrium this yields $\alpha_1 = \alpha_2 = 2r\bar{c}$. □