A Long-run Risks Model with Long- and Short-run Volatilities: Explaining Predictability and Volatility Risk Premium

Guofu Zhou  
Washington University

and

Yingzi Zhu  
Tsinghua University and CCFR

First Draft: October, 2008
Current version: May, 2009

1Correspondence: Guofu Zhou, Olin School of Business, Washington University, St. Louis, MO 63130; e-mail: zhou@wustl.edu, phone: 314-935-6384.
A Long-run Risks Model with Long- and Short-run Volatilities: Explaining Predictability and Volatility Risk Premium

In this paper, we extend the long-run risks model of Bansal and Yaron (2004) as well as the improved version of Bansal, Kiku and Yaron (2007b) by decomposing the volatility of consumption growth, long-run risk, and dividends into two components: a long-run and a short-run volatilities. In doing so, the new model can retain much of the good properties of the original models, and yet it is capable of justifying simultaneously the large negative market variance risk premium, differing predictability in excess returns, consumption and dividends, as well as their volatilities, all of which are difficult to explain previously.
The long-run risks model of Bansal and Yaron (henceforth BY, 2004) is perhaps currently the most viable asset pricing model which successfully explains simultaneously the equity risk premium puzzle, the low risk-free rate puzzle, the high level of market volatility, and many other stylized facts about the stock market, consumption and dividend-price ratio. Consequently, it is not surprising that it has attracted a lot of attention, with important subsequent studies by Bansal, Khatchatrian, and Yaron (2005), Bansal, Kiku, and Yaron (2007a, b), Bansal, Dittmar, and Kiku (2008), Drechsler and Yaron (2008), Hansen, Heaton, and Li (2008), Pakos (2008), Avramov and Hore (2009), and Beeler and Campbell (2009), among others.

However, there remain three major problems with the model. First, the BY model implies a much stronger predictability of consumption growth by dividend-price ratio than observed in the data, as pointed out by Bansal, Kiku and Yaron (henceforth BKY, 2007b). Second, the model implies a variance risk premium that is too small to match that of the options market, as shown by Drechsler and Yaron (2008). Third, the model asserts a much stronger predictive power of future stock return volatility by dividend-price ratio than is found in the data, as demonstrated by Beeler and Campbell (2009). The existing solutions to the first two problems are unsatisfactory. For example, BKY provides an improved version of the model that deals with the first problem, but their approach does not address the second. While Drechsler and Yaron (2008) provide a jump process solution to the second problem, their approach does not apply to fix the first. Moreover the third problem is still an open issue. Overall, the question we ask is: Can the BY model be extended in such a way to overcome all the three problems simultaneously?

In this paper, we extend the BY model by augmenting the improved model of BKY with both a long-run and a short-run volatility components. We decompose the volatility of both the consumption growth and long-run risk into convex combinations of these long-run and short-run volatilities. In doing so, we can retain much of the good properties of the BY and BYK models. By modeling the dividend-growth process in a similar way, we are able to show that the implied stock volatility process is also a linear combination of both the long-run and short-run volatilities, allowing us to easily fit the volatility process and explain the large negative variance risk premium with results consistent with the volatility literature (see, e.g., Christoffersen, Jacobs, Ornthanalai and Wang, 2008, and Lu and Zhu, 2009). In addition, the additional volatility component introduces enough flexibility into the BY and BKY models, so that we can explain also the two remaining
problems of the model, while at the same time can still justify other stylized facts about the stock market, consumption and dividend-price ratio.

In their comprehensive analysis of the BY and BKY models, Beeler and Campbell (2009) provide profound insights and raise a number of important challenges. Nevertheless, it seems that our new model, an extension of the BY and BKY, is capable of meeting all the known and major challenges in the equity market associated with excess stock returns, consumption and dividend-price ratio. The remaining challenge is how well the model performs in the fixed income markets. While this issue will be discussed in detail later in the paper, it may be useful to point out at the outset that this challenge is also the one that speaks to all existing asset pricing models. None of them, as far as we know, is capable of explaining or pricing assets well in both the equity and fixed income markets. For example, the well-known Fama and French (1993) three-factor model explains only the cross-sectional differences in equity expected returns, and has almost nothing to say about the rich and dynamic fixed income markets. On the other hand, the well-known affine models (see, e.g., Dai and Singleton, 2003) can explain the term-structure well with level, slope and curvature factors, but the model is virtually useless in explaining cross-sectional equity expected returns. While Beeler and Campbell’s (2009) emphasis on jointly modelling the equity and fixed income markets is well placed, and it seems to point to perhaps the next most important breakthrough to be made in asset pricing, the long-run risks models still appear important and promising in this direction too, since it provides valuable insights on how to explain well the equity markets first.

From a methodology perspective, our new model is parsimonious. While our framework allows multifactor volatilities, we consider only long- and short-run volatilities, which have easy economic interpretations and are consistent with recent volatility studies by Alizadeh, Brandt, and Diebold (2002), Chernov, Gallant, Ghysels and Tauchen (2003), Chacko and Viceira (2003), Christoffersen, Jacobs, Ornthanalai and Wang (2008), and Lu and Zhu (2009), among others. On the other hand, the two-component volatility extension of the BY and BKY models is not only sufficient in matching the variance risk premium, but also necessary in solving the volatility predictability problem. One-factor volatility model produces volatility predictive slope coefficients that are the same across the variables, hence it will be unable to explain the cross-sectional differences in volatility predictability. In addition, on the technical side, we cast our extension of the BY and BKY models in continuous-time. This allows us not only to avoid the occasional negative volatility problem in the discrete-time
models, but also to obtain approximate analytical solutions for many functions of economic interest, such as derivative prices, measures of volatility and various predictive slope coefficients.

The rest of the paper is organized as follows. Section I provides a short review of the BY model. Section II provides the extension with both the long-run and short-run volatility components, and solves various functions of interest approximate analytically. Section III calibrates the model and examines its implications and ability to explain the major problems facing the BY and BKY models. Section IV discusses the remain challenges and future research. Section V concludes.

I. A Short Review of BY

In this section, we provide a short review of the BY model, which will be useful for understanding our extension and comparisons.

The BY model assumes a representative investor who has Epstein-Zin-Weil preferences (Epstein and Zin 1989, Weil 1989), and makes his optimal portfolio decision under the following discrete-time processes for consumption and dividends:

\[
\begin{align*}
\log(C_{t+1}/C_t) &= \mu_t + X_t + \sigma_t \eta_{t+1}, \\
X_{t+1} &= \alpha X_t + \varphi x \sigma_t e_{t+1}, \\
\sigma_{t+1}^2 &= \sigma^2 + \kappa (\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}, \\
\log(D_{t+1}/D_t) &= \mu_d + \varphi X_t + \varphi_d \sigma_t u_{t+1},
\end{align*}
\]

(1)

where \( X_t \) is the long-run risk that affects both consumption and dividend growth, \( \log(C_{t+1}/C_t) \) and \( \log(D_{t+1}/D_t) \), and is persistent with autoregression (AR) coefficient \( \alpha \) and volatility \( \varphi_x \sigma_t \); the variance process \( \sigma_t^2 \) is also a time-varying AR process with coefficient \( \kappa \); \( \eta_{t+1}, e_{t+1}, w_{t+1} \) and \( u_{t+1} \) are independent shocks drawn from the standard normal distribution.

The intuition of the model is that \( X_t \) captures the long-run growth prospects of the economy. Shocks in both long-run \( X_t \) and short-run \( \eta_{t+1} \) drive the consumption growth and asset prices. The fear for adverse long-run growth requires high risk premium to compensate. Along with the long-run and short-run shocks in dividend growth, asset prices can be very volatile. As a result,
the BY model can successfully explain equity risk premium, the risk-free rate, the volatility of the market return, and the predictability of the price-dividend ratio on the stock market returns, among other stylized facts of the equity market.

However, there are some strong implications from the original BY model that are inconsistent with market data. Empirically, price-dividend ratio has little power in predicting consumption growth, but the model implies that it predicts consumption growth as well as the excess stock returns. BKY address this issue by increasing the persistence of the volatility to make it a more important factor in price-dividend ratio to reduce the importance of long-run consumption risk factor, hence the predictability of consumption growth. However, as demonstrated later, the increase in the persistence of the volatility entails a too small (virtually zero) variance risk premium. Drechsler and Yaron (2008) allow jumps in the volatility process that is capable of explaining the large negative variance risk premium, but their approach does not apply to explain the volatility predictability problem below. In addition, multifactor stochastic volatility models is generally superior in capturing volatility term structure dynamics (see, e.g., Christoffersen, Jacobs, Ornthanalai and Wang, 2008, and Lu and Zhu, 2009). The multifactor stochastic volatility approach is what we use below, and, as it turns out, this approach accommodates various predictability properties of the long-run risks model as well. Moreover, it can also help to address the volatility predictability problem pointed out by Beeler and Campbell (2009). Overall, despite the huge success of the BY and BKY models elsewhere, their extensions are called for to resolve the variance risk premium problem and various predictability problems.

II. The New Long-Run Risks Model

In this section, we first motivate our dynamic processes for the state variables of the new long-run risks model, and then solve the model. Subsequently, we provide approximate analytical solutions to functions of interest: the consumption-wealth ratio, market prices of risks, price-dividend ratio, and the market return volatility.
A. The Model and Solution

Our model is an extension of the BY and BKY models and is cast in the continuous-time framework. In this subsection, we first provide a generalized version of Equation (1) with long- and short-run volatilities, and then solve essentially the value function of the representative investor’s optimization problem, which lays the foundation for solution to the other functions of interest.

Parallel to the discrete-time model (1), we consider the following model for the consumption and dividends processes and their related variables,

\[
\frac{dC_t}{C_t} = (\mu + X_t)dt + \sqrt{V_1 t \delta_c^2 + V_2 t (1 - \delta_c^2)} dZ_{1t} \\
\frac{dX_t}{X_t} = -\alpha X_t dt + \varphi_x \sqrt{V_1 t \delta_x^2 + V_2 t (1 - \delta_x^2)} dZ_{2t} \\
\frac{dD_t}{D_t} = (\mu_d + \varphi X_t) dt + \varphi_d \sqrt{V_1 t \delta_d^2 + V_2 t (1 - \delta_d^2)} dB_t + \sigma_{dc} \sqrt{V_1 t \delta_c^2 + V_2 t (1 - \delta_c^2)} dZ_{1t} \\
\quad \quad \quad + \sigma_{dx} \sqrt{V_1 t \delta_x^2 + V_2 t (1 - \delta_x^2)} dZ_{2t} + \sigma_{dv} \sqrt{V_1 t} dw_{1t} \\
\frac{dV_{1t}}{V_{1t}} = \kappa_1 (\bar{V}_1 - V_{1t}) dt + \sigma_1 \sqrt{V_{1t}} dw_{1t} \\
\frac{dV_{2t}}{V_{2t}} = \kappa_2 (\bar{V}_2 - V_{2t}) dt + \sigma_2 \sqrt{V_{2t}} dw_{2t}, \quad \kappa_1 < \kappa_2,
\]

where \(dZ_{1t}, dZ_{2t}, dB_t, dw_{1t}\) and \(dw_{2t}\) are independent shocks drawn from the standard Brownian motion. When \(\delta_x = 1, \delta_c = 1, \delta_d = 1, \) and \(\sigma_{dc} = \sigma_{dx} = \sigma_{dv} = 0,\) the above model reduces to the continuous-time limit of the BY model. When \(\delta_x = 1, \delta_c = 1, \delta_d = 1, \) and \(\sigma_{dc} = 0,\) it becomes the continuous-time limit of the BKY model. Therefore, the above model is a continuous-time extension of both the BY and BKY models that nest the original ones as special cases.

The key feature of the new model is that the consumption growth has a volatility level of \(V_1 t \delta_c^2 + V_2 t (1 - \delta_c^2),\) a convex combination of the long- and short-run volatilities \(V_{1t}\) and \(V_{2t}.\) This convex combination not only decomposes the total volatility into two plausible components, but also allows for easy solutions below. The same volatility decomposition, adjusted for the leverage factor \(\varphi_x,\) is also applied to the long-run risk \(X_t.\) The dividend growth process is treated similarly, except that it allows for various covariations with \(C_t\) and \(X_t,\) as in the BKY model. Finally, the long- and short-run volatilities follow the standard square-root Heston process.

To solve the equilibrium prices and other quantities of interest, following the BY model, we also use the Epstein-Zin-Weil preference, but in continuous-time. Based on Duffie and Epstein (1992),
we can define the intertemporal value function recursively by

$$J_t = E_t[\int_t^T f(C_s, J_s)ds].$$

(3)

Then the representative investor’s objective is to choose consumption and security holdings to

$$\max_{\{C_t\}} E_0[\int_0^T f(C_s, J_s)ds],$$

(4)

where $f(C, J)$ is a normalized aggregator related to current consumption $C_t$ and continuation value function $J_t$, and is given by

$$f(C, J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J[(\frac{C}{(1-\gamma)J})^{1-\frac{1}{\psi}} - 1],$$

(5)

with $\beta$ the rate of time preference, $\gamma > 0$ the relative risk aversion, and $\psi > 0$ the elasticity of intertemporal substitution (IES). It is well known that if we set $\psi = 1/\gamma$ in (5), we obtain the standard additive expected utility of constant relative risk aversion (CRRA).

Following Duffie and Epstein (1992), we have the pricing kernel

$$\pi_t = \exp\left[\int_0^t f(J(C_s, J_s)ds \right] f_C(C_t, J_t)$$

(6)

with $J = J(c, x, v_1, v_2)$, a solution of

$$f(c, J) + cf_c(\mu + c) + \frac{1}{2}[\delta_C^2 v_1 + (1-\delta_C^2) v_2] C^2 J_{cc} + J_c \cdot (\alpha c) + \frac{1}{2}[\delta_C^2 v_1 + (1-\delta_C^2) v_2] J_{xx}$$

$$+ J_{v_1} \cdot \kappa_1 (\bar{V}_1 - v_1) + \frac{1}{2} \sigma_{11}^2 v_1 J_{v_1 v_1} + J_{v_2} \cdot \kappa_2 (\bar{V}_2 - v_2) + \frac{1}{2} \sigma_{22}^2 v_2 J_{v_2 v_2} = 0.$$  

(7)

Let $g_1$ be the long-term mean of the consumption-wealth ratio,$^1$

$$g_1 = \left(\frac{C_t}{W_t}\right) = \exp(c_t - \omega_t),$$

(8)

where the lowercase variables are the log variables. With the standard log-linear approximation of Campbell (1993) who develops it first in discrete time, and which Chacko and Viceira (2005) use first in continuous time, we have

$$\frac{C_t}{W_t} = \exp(c_t - \omega_t) \approx g_1 - g_1 \log g_1 + g_1 \log(C_t/W_t).$$

(9)

Conjecturing a solution for $J$ of the following form,

$$J(W_t, X_t, V_{1t}, V_{2t}) = \exp(A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) \frac{W_t^{1-\gamma}}{1-\gamma},$$

(10)

$^1$It can be solved endogenously once the model parameters are known. Appendix A.3 provides the details.
we can solve the Bellman equation (7) to obtain (see Appendix A.1)

\[
A_0 = \frac{1}{g_1 \psi} \left[ \theta \xi + (1 - \gamma) \mu + \kappa_1 \tilde{V}_1 \psi A_2 + \kappa_2 \tilde{V}_2 \psi A_3 \right]
\]

\[
A_1 = \frac{1 - \gamma}{(g_1 + \alpha) \psi}
\]

\[
A_2 = -\frac{b_1 - \sqrt{b_1^2 - 4a_1 c_1}}{2a_1}
\]

\[
A_3 = -\frac{b_2 - \sqrt{b_2^2 - 4a_2 c_2}}{2a_2}
\]

with

\[
a_1 = \frac{1}{2} \sigma_1^2 \psi^2, \quad b_1 = -(g_1 + \kappa_1) \psi, \quad c_1 = -\frac{1}{2} \gamma (1 - \gamma) \hat{\delta}_c^2 + \frac{1}{2} \varphi_x \hat{\delta}_x^2 \frac{(1 - \gamma)^2}{(g_1 + \alpha)^2}
\]

\[
a_2 = \frac{1}{2} \sigma_2^2 \psi^2, \quad b_2 = -(g_1 + \kappa_2) \psi, \quad c_2 = -\frac{1}{2} \gamma (1 - \gamma) \hat{\delta}_c^2 (1 - \hat{\delta}_c^2) + \frac{1}{2} \varphi_x \hat{\delta}_x^2 (1 - \hat{\delta}_x^2) \frac{(1 - \gamma)^2}{(g_1 + \alpha)^2}.
\]

The solution is approximate in general and exact when \( \psi = 1 \). Armed with this solution, we are ready to solve for various functions of interest, which are also approximate in general and exact when \( \psi = 1 \).

**B. Consumption-wealth Ratio**

Consider first the consumption-wealth ratio. Based on (A6) and its proof, we immediately have

\[
\frac{C_t}{W_t} = \beta \psi \exp \{ (A_{0a} + A_{1a} X_t + A_{2a} V_{1t} + A_{3a} V_{2t}) \},
\]

where \( A_{ia} = A_i \frac{1 - \psi}{1 - \gamma} \) for \( i = 0, 1, 2, 3 \). The ratio is loglinear in the state variables, and has similar functional form as in the BY model. In particular,

\[
A_{1a} = -\frac{1}{g_1 + \alpha},
\]

which is exactly the same as the continuous analogue of their \(-A_1\).\(^3\) Hence, the same interpretation applies that, when \( \psi < 1 \) the income effect dominate, and when \( \psi > 1 \), the substitution effect dominates. In addition, the consumption-wealth ratio is more sensitive to the expected growth rate when the persistence of expected growth shocks, measured by \( 1 / \alpha \), increases.

\(^2\)As found in similar approximations elsewhere as well as our own studies, the approximate solution is accurate around commonly calibrated parameters.

\(^3\)Note that Bansal and Yaron (2004) use the ratio of wealth to consumption, but we use the ratio of consumption to wealth. Hence our \( A_{ia} \)'s have the opposite sign of theirs. The same applies to the price-dividend ratio below.
However, there are now two volatilities as expected. Due to their symmetric formulation entered into the consumption and dividends dynamics, the two volatility components impact on the ratio in the same way, with $A_{2a}$ and $A_{3a}$ depending on the volatility parameters via similar functional forms. When $\psi > 1$, both $A_{2a}$ and $A_{3a}$ are positive. The same intuition of the BY about volatility also holds here. For example, a rise in either of the volatilities will make consumption more volatile, which lowers asset valuations and increases the risk premia on all assets. In addition, an increase in the persistence of volatility shocks, that is, a decrease in either $\kappa_1$ or $\kappa_2$, will magnify the effects of volatility shocks on valuation ratios, since the investor would perceive changes in economic uncertainty as being long lasting.

C. Risk-free Rate and Market Prices of Risks

Recall that the pricing kernel is given by Equation (6). Based on the solution for $f$, we have

$$ f_J = \xi_1 - g_1(A_1X_t + A_2V_{1t} + A_3V_{2t}) \frac{1 - \gamma \psi}{1 - \gamma} $$

$$ f_C = \beta^{\psi \gamma} \exp \left[(B + A_1X_t + A_2V_{1t} + A_3V_{2t}) \frac{1 - \gamma \psi}{1 - \gamma} \right] C_t^{-\gamma}, $$

where

$$ \xi_1 = (\theta - 1)\xi - \beta - g_1 \frac{1 - \gamma \psi}{1 - \gamma} B. $$

(16)

Now applying Ito’s Lemma to $\pi_t$, we have

$$ \frac{d\pi_t}{\pi_t} = -(r_f dt + \lambda_1 dZ_{1t} + \lambda_2 dZ_{2t} + \lambda_3 dw_{1t} + \lambda_4 dw_{2t}), $$

(17)

where the risk-free rate $r_f$ and the market prices of risks, $\lambda_i, i = 1, 2, 3, 4$, are given below.

First, the risk-free rate is

$$ r_f = -(r_0 + r_1X_t + r_2V_{1t} + r_2V_{2t}), $$

(18)

where

$$ r_0 = \xi_1 + (\kappa_1A_2\bar{V}_1 + \kappa_2A_3\bar{V}_2) \frac{1 - \gamma \psi}{1 - \gamma} - \gamma \mu $$

$$ r_1 = -\frac{1}{\psi} $$

$$ r_2 = -(g_1 + \kappa_1)A_2 \frac{1 - \gamma \psi}{1 - \gamma} + \frac{1}{2} \left( \frac{1 - \gamma \psi}{1 - \gamma} \right)^2 (A_1^2\varphi_x^2\delta_x^2 + A_1^2\sigma_1^2) + \frac{1}{2} \gamma (\gamma + 1) \delta_c^2 $$

(19)

$$ r_3 = -(g_1 + \kappa_2)A_3 \frac{1 - \gamma \psi}{1 - \gamma} + \frac{1}{2} \left( \frac{1 - \gamma \psi}{1 - \gamma} \right)^2 [A_1^2\varphi_x^2(1 - \delta_x^2) + A_3^2\sigma_2^2] + \frac{1}{2} \gamma (\gamma + 1)(1 - \delta_c^2). $$
Note that $r_1 < 0$, implies that the risk-free rate increases with increasing expectation of growth. Further, when $\gamma > 1/\psi$, since $A_2$ and $A_3$ are all positive, both $r_2$ and $r_3$ are positive, implying that the risk-free rate decreases as the consumption uncertainty increases.

The market prices of risks are

$$\begin{align*}
\lambda_1 &= \gamma \sqrt{V_{1t}\delta^2_z + V_{2t}(1 - \delta^2_c)} \\
\lambda_2 &= -\frac{1 - \gamma \psi}{1 - \gamma} A_1 \varphi x \sqrt{V_{1t}\delta^2_x + V_{2t}(1 - \delta^2_x)} \\
\lambda_3 &= -\frac{1 - \gamma \psi}{1 - \gamma} A_2 \sigma_1 \sqrt{V_{1t}} \\
\lambda_4 &= -\frac{1 - \gamma \psi}{1 - \gamma} A_3 \sigma_2 \sqrt{V_{2t}}.
\end{align*}$$  \hspace{1cm} (20)

As usual, the magnitude of the risk aversion relative to reciprocal of the IES determines whether agents prefer early or late resolution of uncertainty regarding consumption path. When agents prefer early resolution of uncertainty, that is, when $\gamma > 1/\psi$, the market prices for long run risk is positive. In addition, since $A_1$, $A_2$ and $A_3$ are all increasing in magnitude as the persistence parameters, $1/\alpha$, $1/\kappa_1$ and $1/\kappa_2$ increase, consequently, risk premia increase.

Moreover, the market prices of risks for variances are negative when $\gamma > 1/\psi$. This is consistent with the empirical evidence that the risk premium for variance is negative. Intuitively, the negative sign on the variance risk premium indicates that investors regard increases in market volatility as unfavorable shocks to the investment opportunity. However, in contrast with the BY and BKY models, the market variance risk premium is determined by both the long- and short-run volatilities. As will be clear later, because of the rich dynamics of these two components, the associated parameters can be chosen in such a way to explain the market variance risk premium, while the previous models do not allow for such flexibility.

D. Price-dividend Ratio

With the market prices of risks, we can solve the price-dividend ratio as follows:

$$\frac{D_t}{P_t} = \exp\{ (A_{0m} + A_{1m} X_t + A_{2m} V_{1t} + A_{3m} V_{2t}) \},$$  \hspace{1cm} (21)

where

$$A_{1m} = \frac{\varphi - \frac{1}{\psi}}{g_{1m} + \alpha}. \hspace{1cm} (22)$$
and the other $A_{im}$’s are given in Appendix A.2, with $g_{im}$, similar to $g_1$, as the long-term mean of $D_t/P_t$. Note that when $\varphi > 1$, $|A_{1m}| > |A_{1a}|$, consequently, expected growth rate news leads to a larger reaction in the price of the dividend claim than in the price of the consumption claim. In addition, when $\varphi$ is big enough, $A_{2m}$ and $A_{3m}$ are positive, resulting in the well-known leverage effect, i.e., the shocks to return is negatively correlated with shocks to variance process.

E. The Market Volatility

Applying Ito’s Lemma to

$$P_t = D_t \exp\{- (A_{0m} + A_{1m}X_t + A_{2m}V_{1t} + A_{3m}V_{2t})\},$$

we obtain

$$\frac{dP_t}{P_t} = \left[ \mu_t - (A_{2m}\kappa_1\bar{V}_1 + A_{3m}\kappa_2\bar{V}_2) + (\varphi + \alpha A_{1m})X_t + A_{2m}\kappa_1V_{1t} + A_{3m}\kappa_2V_{2t} \right] dt$$

$$+ \varphi d\sqrt{V_{1t}\delta^2_d + V_{2t}(1 - \delta^2_d)}dB_t + \sigma_{dc}\sqrt{V_{1t}\delta^2_c + V_{2t}(1 - \delta^2_c)}dZ_{1t}$$

$$+ \sigma_{dx}\sqrt{V_{1t}\delta^2_x + V_{2t}(1 - \delta^2_x)}dZ_{2t} + \sigma_{dv}\sqrt{V_{1t}dw_{1t}}$$

$$- A_{1m}\varphi x\sqrt{V_{1t}\delta^2_x + V_{2t}(1 - \delta^2_x)}dZ_{2t}$$

$$- A_{2m}\sigma_1\sqrt{V_{1t}dw_{1t}} - A_{3m}\sigma_2\sqrt{V_{2t}dw_{2t}}$$

$$= \left[ \mu_t - (A_{2m}\kappa_1\bar{V}_1 + A_{3m}\kappa_2\bar{V}_2) + (\varphi + \alpha A_{1m})X_t + A_{2m}\kappa_1V_{1t} + A_{3m}\kappa_2V_{2t} \right] dt + \sqrt{V_{1t}}dZ_t$$

where $dZ_t$ is another Brownian motion, and hence the variance of the price process is

$$V_t = c_1V_{1t} + c_2V_{2t} \tag{23}$$

with

$$c_1 = \phi^2\delta^2_d + \sigma^2_{dc}\delta^2_c + (\sigma_{dx} - A_{1m}\varphi x)^2\delta^2_x + (\sigma_{dv} - A_{2m}\sigma_1)^2 \tag{24}$$

$$c_2 = \phi^2(1 - \delta^2_d) + \sigma^2_{dc}(1 - \delta^2_c) + (\sigma_{dx} - A_{1m}\varphi x)^2(1 - \delta^2_x) + A^2_{2m}\sigma^2_2 \tag{25}$$

III. Calibration

In this section, based on the BY, BKY and Beeler and Campbell (2009), we first calibrate the model parameters and examine their moment matching properties. Then we discuss predictability of excess returns, consumption and dividends growth, as well as their volatilities. Finally, we analyze the risk premium on the market variance.
A. Calibrated Parameters and Moment Matching

To compare the results, we first translate the parameters of the BY and BKY models into their continuous-time counterparts. With the monthly estimates provided by Bansal and Yaron (2004) and Beeler and Campbell (2009), we can annualize them and match the discrete- and continuous-time volatility processes. In this way, all the parameters of the continuous-time models corresponding to either the BY or the BKY model can be obtained. We report the results in Table I.

To retain the good and desired properties of the BY and BKY models, we set most of the common parameters of our two-factor volatility model at roughly the same values with their one-factor models, while calibrate the additional parameters to meet various requirements. This will be clear as we go along in discussing the various results below. On the preference parameters, Bansal and Yaron (2004) use both $\gamma = 7.5$ and 10. We use, however, a lower value of $\gamma = 6$. Theoretically, this is a more plausible level of risk aversion. However, it will imply a lower market risk premium, which will be discussed further below. On parameters governing the consumption growth, all the parameters, $\mu, \alpha$ and $\varphi_x$, are virtually the same across the models (the $\delta$'s are additional parameters). For the common parameters in the dividend growth process, the new model’s calibrations are somewhat different, but not by much. For the common volatility parameters, we calibrate our parameters closely to the BKY model. The volatility of the BKY is much more persistent than the BY model. Although $\kappa_1$ in our new model has almost the same value as BKY, the total volatility is not as persistent as in BKY, because there are two components in our model, and the large size short-run volatility component is not persistent at all. These parameter values in our model imply reasonable volatility components for the market returns that are consistent with the volatility literature (see, e.g., Christoffersen, Jacobs, Ornthanalai and Wang, 2008, and Lu and Zhu, 2009).

The moments of the asset pricing variables computed from monthly data from February 1947 to March 2007, as well as those implied by all the three models are presented in Table II. Those of the data and of the BY and BKY models are based on Beeler and Campbell (2009) with $\gamma = 10$, and those of our new model are evaluated by the analytical formulas of the previous section with the calibrated parameters in Table I. Note that the moments of our new model are largely in agreement with the data and with the BY and BKY, with two notable exceptions. First, the market risk

---

4Technically, this smaller value is useful to ensure the Heston volatility models are well behaved.
premium is 3.58%, lower than about 6% from the data, and lower than about 6.6% from both the BY and BKY models. As mentioned earlier, this is due to our choice of a lower $\gamma = 6$. The question is whether 3.58% is a reasonable market risk premium level. According to Jagannathan, McGrattan and Scherbina (2001), and Fama and French (2002), the equity risk premium is declining, and the former article even states that “the premium averaged about 7 percentage points during 1926–70 and only about 0.7 of a percentage point after that.” Given these findings and the recent financial crisis, a 3.58% risk premium does not seem too low. The second exception is the volatility of the risk-free rate in our model is higher than those of the BY and BKY models, but this is in fact desired as it matches the data better. Overall, the new factor model matches the moments of the data nicely, and does so as well as the BY and BKY models.

B. Predictability of Excess Returns, Consumption and Dividends

To examine predictability of the variables, we, following Beeler and Campbell (2009), consider the following three $K$-period regressions

$$
(r_{t+j} - r_{f,t+j}) + \cdots + (r_{t+j+K} - r_{f,t+j+K}) = \alpha_{jK} + \beta(p_t - d_t) + \epsilon_{jKt} 
$$

(26)

$$
\Delta c_{t+j} + \cdots + \Delta c_{t+j+K} = \alpha_{jK} + \beta(p_t - d_t) + \epsilon_{jKt} 
$$

(27)

and

$$
\Delta d_{t+j} + \cdots + \Delta d_{t+j+K} = \alpha_{jK} + \beta(p_t - d_t) + \epsilon_{jKt}, 
$$

(28)

where $r$ and $r_f$ are the market return and risk-free rate, respectively; and $c_t$ and $d_t$ are logarithms of consumption and dividends. To explain the observed regression patterns, our idea is to derive the regression slope coefficients as functions of the model parameters. Then, given that our new model has more, we can choose them in such a way to make the model implied regression slope coefficients match closely with those of the data.

We provide the formulas for $K = 1$ only for notational simplicity, while the general case is a straightforward extension. As $K = 1$, the regressors of the above three regressions all have the same functional form of

$$
dY_t = [a_0 + a_1 X_t + a_2 V_{1t} + a_3 V_{2t}]dt + \sqrt{b_1 V_{1t} + b_2 V_{2t}}dZ_t, 
$$

(29)
where \(dY_t\) corresponds to \(d \ln P_t - r_f dt\), \(d \ln C_t\) and \(d \ln D_t\), respectively, with
\[
a_1 = \varphi + \alpha A_{1m} + r_1 + g_{1m} A_{1m}, \quad a_2 = A_{2m} \kappa_1 - \frac{b_1}{2} + r_2 + g_{1m} A_{2m}, \quad a_3 = A_{3m} \kappa_2 - \frac{b_2}{2} + r_3 + g_{1m} A_{3m}
\]
in the first case,
\[
a_1 = 1, \quad a_2 = -\frac{\delta^2}{2}, \quad a_3 = -\frac{1 - \delta^2}{2}
\]
in the second case, and
\[
a_1 = \varphi, \quad a_2 = -\varphi \gamma_2 + \frac{\sigma^2 \alpha}{2} \gamma_2^2 + \frac{\sigma^2 \alpha}{2} \gamma_2^2 + \frac{\sigma^2 \alpha}{2} \gamma_2^2, \quad a_3 = -\varphi \gamma_2 (1 - \delta^2) + \frac{\sigma^2 \alpha}{2} (1 - \delta^2) + \frac{\sigma^2 \alpha}{2} (1 - \delta^2)
\]
in the last one. The regression slope coefficient is then analytically obtained as
\[
b = \frac{\text{Cov}(\Delta y, p - d)}{\text{Var}(p - d)}, \quad (30)
\]
where
\[
\text{Cov}(\Delta y, p - d) = -\left[ a_1 A_{1m} \frac{\sigma^2}{2 \alpha} (1 - e^{-\alpha \tau}) + a_2 A_{2m} \frac{\sigma^2 \bar{V}_1}{2 \kappa_1} (1 - e^{-\kappa_1 \tau}) + a_3 A_{3m} \frac{\sigma^2 \bar{V}_2}{2 \kappa_2} (1 - e^{-\kappa_2 \tau}) \right], \quad (31)
\]
\[
\text{Var}(p - d) = A_{1m}^2 \frac{\sigma^2}{2 \alpha} + A_{2m}^2 \frac{\sigma^2 \bar{V}_1}{2 \kappa_1} + A_{3m}^2 \frac{\sigma^2 \bar{V}_2}{2 \kappa_2} \quad (32)
\]
with
\[
\sigma^2 = \varphi^2 \left[ \bar{V}_1 \delta^2 + \bar{V}_2 (1 - \delta^2) \right]. \quad (33)
\]
The proof is given in Appendix A.4.

Table III reports the calibrated results, which are in the last column of the table. The first three columns are the regression betas from the data, the BY and the BKY models, all of which are taken from Beeler and Campbell (2009).\(^5\) On the predictability of excess stock returns by the price-dividend ratio, the data have betas ranging from \(-0.059\) to \(-0.421\) as horizon increases from 1 year to 5 years, but the BY model does not match this with betas from \(-0.007\) to \(-0.039\). In contrast, the BKY does the matching well with betas from \(-0.078\) to \(-0.368\). In line with the BKY results, our new model performs as well with betas from \(-0.074\) to \(-0.351\).

Opposite to its low predictability on excess stock returns, the BY model forecasts consumption growth with betas from 0.114 to 0.338, which are substantially higher than those from data, which

---

\(^5\)We do not report \(R^2\) for brevity, since the \(R^2\)s must match once the second moments and betas are matched.
are lower than 0.01 in absolute values. The BY calibration also has too much power in predict dividend growth. In contrast, the BKY model and our new one match the data fairly well on both fronts. Overall, our model inherits much of many good properties of the BKY model, and so it performs as well in explaining the predictability of excess returns, consumption and dividends.

However, the BKY calibration inevitably increases the predictability of volatility because of the increased importance of the volatility in the price-dividend ratio. In addition, as we show later, as long as there is only one volatility component, the predictive regression slope coefficients of log volatility for the three variables (excess return, consumption growth, dividend growth) should be the same. Therefore, we must go beyond the BKY model to understand the differing predictability of volatility, as well as the large negative variance premium, which are addressed in the next two subsections.

C. Predictability of Volatility: Excess Returns, Consumption and Dividends

Following Beeler and Campbell (2009), we use the realized volatility measure. There are two steps in computing this measure. First, we run an AR(1) regression of each variable of interest \( y_{t+1} \),

\[
y_{t+1} = b_0 + b_1 y_t + u_{t+1},
\]

where \( y_{t+1} \) is the excess return or consumption growth or dividend growth. Second, the \( K \)-period realized volatility is defined as the sum of the absolute values of the residuals,

\[
\text{Vol}_{t:t+K-1} = \sum_{k=0}^{K-1} |u_{t+k}|
\]

over \( K \) periods with \( K \), as before, the horizon of interest.

Then, the predictability of volatility is examined from the regression of the log of \( K \)-period realized volatility on the log price-dividend ratio,

\[
\ln[\text{Vol}_{t+1:t+K}] = \alpha + \beta(p_t - d_t) + \xi_t.
\]

To match the volatility predictability of the data, we need to solve the continuous-time version of the regression slope.

First, we note that the innovation process \( u_{t+1} \) in the AR(1) process (34) is a discrete-time version of \( dZ_t \) in equation (29). Thus, in the continuous limit, the discrete realized volatility
defined in (35) can be written as

$$\text{Vol}_{t,t+\tau} = \int_0^\tau \sqrt{V_t} |dZ_t| = \frac{2}{\sqrt{2\pi}} \int_0^\tau \sqrt{V_t} dt,$$

(37)

where we have used the relation

$$|dZ_t| = \frac{2}{\sqrt{2\pi}} dt$$

for a standard Brownian Motion $Z_t$. Then, a key step is in approximating the log of $\tau$ period realized volatility is to use the following approximate equality,

$$\ln \int_0^\tau \sqrt{V_t} |dZ_t| \approx \text{Const} + \frac{1}{2\tau} \int_0^\tau \frac{V_t}{V} dt,$$

(38)

where $V$ is the unconditional mean of $V_t$ (see Appendix A.5). Then, we can obtain approximately the volatility regression coefficient as

$$\beta = \frac{\text{Cov}(\Delta \tau y, p - d)}{\text{Var}(p - d)},$$

(39)

where

$$\text{Cov}(\Delta \tau y, p - d) = - \left[ b_1 A_2 m \frac{\sigma_1^2 V_1}{2\kappa_1^2} (1 - e^{-\kappa_1 \tau}) + b_2 A_3 m \frac{\sigma_2^2 V_2}{2\kappa_2^2} (1 - e^{-\kappa_2 \tau}) \right],$$

(40)

$\text{Var}(p - d)$ is the same as given by equation (32), and $b_1$ and $b_2$ are given by

$$b_1 = c_1, \quad b_2 = c_2$$

in the case of excess return volatility,

$$b_1 = \delta^2_c, \quad b_2 = 1 - \delta^2_c$$

in the case of consumption growth volatility, and

$$b_1 = \varphi^2 d + \sigma^2 d \delta^2_c + \sigma^2 d \delta^2_c + \sigma^2 d, \quad b_2 = \varphi^2 (1 - \delta^2_d) + \sigma^2 d (1 - \delta^2_c) + \sigma^2 d (1 - \delta^2_c)$$

in the final case of dividend growth volatility.

The approximate equality (38) is not only important for obtaining the volatility slope regression coefficient, but also important for understanding why the one-factor volatility models of the BY and BKY cannot explain the cross-sectional predictability in volatility. If the variance process $V_t$
has only one factor, the volatilities of excess return, consumption and dividends all must be a linear function of this factor. Since
\[
\frac{1}{2\tau} \int_{0}^{\tau} \frac{V_t}{\bar{V}} dt
\]
is invariant to the same scaling on both \(V_t\) and \(\bar{V}\), it follows that the approximated regressors in equation (38) will be the same for any of the three volatility variables. Hence, the volatility regression slope coefficients for all of the three volatility variables must be roughly the same across the variables in either the BY or the BKY models. This is indeed the case as also shown by Beeler and Campbell (2009) in their comprehensive calibration studies.

However, the volatility regression slope coefficients based on data, taken from Beeler and Campbell (2009), show substantial differences across the three variables. For example, as reported in Table IV which are based on Beeler and Campbell (2009) except the last column, \(\beta\) is only \(-0.081\) for excess return volatility, but a much sizeable value of \(-0.530\) for dividend volatility, when a one-year period is considered. Therefore, in order to match the empirical evidence on volatility predictability, it is necessary to incorporate at least one more volatility factor in the BY or BKY models.

Indeed, in our proposed two-factor model with calibrated parameters reported in Table I, the volatility regression slope coefficients match remarkably well with those from the data. With varying time periods for excess return volatility, while the data imply decreasing beta values of \(-0.081\), \(-0.59\) and \(-0.017\), so does the model with \(-0.133\), \(-0.064\) and \(-0.050\). For the consumption volatility, the betas are \(-0.481\), \(-0.491\) and \(-0.564\) with no apparent patterns of either increasing or decreasing, the model provides \(-0.494\), \(-0.446\) and \(-0.431\). Finally, the match in dividends volatility is also almost perfect. In short, additional volatility factor is a key to explain the differences in volatility predictability across excess returns, consumption and dividends. As it turns out, that is also fundamentally important in explaining the market variance premium, as discussed next.

**D. Variance risk premium**

First, it is easy to see that, in our model, the risk premia associated with \(V_{1t}\) and \(V_{2t}\) are
\[
\begin{align*}
\lambda_3 \sigma_1 \sqrt{V_{1t}} &= -\nu_1 V_{1t} \\
\lambda_4 \sigma_2 \sqrt{V_{2t}} &= -\nu_2 V_{2t},
\end{align*}
\]
where 
\[
\nu_1 = \frac{1 - \gamma \psi}{1 - \gamma} A_2 \sigma_1^2 \\
\nu_2 = \frac{1 - \gamma \psi}{1 - \gamma} A_3 \sigma_2^2.
\]

(42)

The risk-neutral processes for \( V_1_t \) and \( V_2_t \) are
\[
dV_1 = \kappa_1^Q \left( \frac{\kappa_1}{\kappa_1^Q} \tilde{V}_1 - V_1 \right) dt + \sigma_1 \sqrt{V_1} dw_{1t}^Q \\
dV_2 = \kappa_2^Q \left( \frac{\kappa_2}{\kappa_2^Q} \tilde{V}_2 - V_2 \right) dt + \sigma_2 \sqrt{V_2} dw_{2t}^Q,
\]

(43)

where
\[
\kappa_1^Q = \kappa_1 - \nu_1 \\
\kappa_2^Q = \kappa_2 - \nu_2.
\]

Then we can show that
\[
\nu_1 < \kappa_1 \text{ and } \nu_1 < \kappa_2
\]

to make the risk-neutral processes stationary. Otherwise, they will blow up in finite time.

Now, it is of interest to see why the one-factor volatility models of the BY and the BKY cannot explain the market variance risk premium. Their \( \kappa \)'s are kept small because of high persistence of consumption growth. For example, with a value of \( \kappa = 0.0139 \) of the BKY parameter calibration, the monthly variance risk premium in absolute value will be bounded from above by \( 0.0139 \times 0.04/12 = 0.463 \) if \( V_t = 0.20^2 = 0.04 \) for market price variance,\(^6\) which is much smaller than that from data, as reported in Table V. However, with an additional volatility component that is less persistent, the new model can produce the desired variance risk premium while keeping most of the moments of the model intact.

Drechsler and Yaron (2008) seems the first to explain the negatively large market variance premium by introducing a jump process into the volatility of the long-run risks model, while stochastic volatility and the long-run risks general equilibrium model are analyzed by Banal, Gallant and Tauchen (2007), and Braker and Shaliastovich (2008), among others, and there is a large volatility literature that focuses on using reduced form models. Following Drechsler and Yaron

---

\(^6\)The variance premium is multiplied by 10000. We use this scaling and the monthly values following Drechsler and Yaron (2008).
(2008), we measure the market variance premium from data as the difference between the squared VIX index and realized variance that captures attitudes toward uncertainty in the equity market. Theoretically, the implied market variance premium is computed in three steps. First, we compute the squared VIX, or more generally, variance swap rate $V S_t$ with maturity $\tau_0$, which is a linear function of the instantaneous variance,

$$V S_t = \sum_{i=1}^{2} (A_i^Q + B_i^Q V_{it}),$$  \hspace{1cm} (45)

where $A_i^Q$ and $B_i^Q$ ($i = 1, 2$) are constants given by

$$A_i^Q = \frac{\kappa_i \bar{V}_i}{\kappa_i^Q} \left[ 1 - \frac{1 - e^{-\kappa_i^Q \tau_0}}{\kappa_i^Q \tau_0} \right], \quad B_i^Q = \frac{1 - e^{-\kappa_i^Q \tau_0}}{\kappa_i^Q \tau_0}.$$  \hspace{1cm} (46)

Second, we evaluate the realized variance,

$$RV_t = \sum_{i=1}^{2} (A_i^P + B_i^P V_{it}),$$  \hspace{1cm} (47)

where $A_i^P$ and $B_i^P$ ($i = 1, 2$) are constants given by

$$A_i^P = \bar{V}_i \left[ 1 - \frac{1 - e^{-\kappa_i \tau_0}}{\kappa_i \tau_0} \right], \quad B_i^P = \frac{1 - e^{-\kappa_i \tau_0}}{\kappa_i \tau_0}.$$  \hspace{1cm} (48)

Finally, the difference between realized variance (46) and variance swap rate (45) is the model implied variance risk premium on the variance swap,

$$VRP = \sum_{i=1}^{2} [(A_i^P - A_i^Q) + (B_i^P - B_i^Q) V_{it}].$$

Table V reports the results, which are measured monthly and multiplied by 10000. The second and third columns, based on the data and the Drechsler and Yaron (2008) model, are taken from their paper directly. The rest are results for the BY, the BKY and our new model, which are computed based on equation (48) with the calibrated parameters. While the data show a risk premium of the level of $-11.27$, both the BY and BKY imply a levels of only $-0.005$ and $-0.010$, too small to explain anywhere close to the true range. In contrast, both Drechsler and Yaron (2008) and our new model imply $-7.57$ and $-6.04$, respectively, which are much closer to the empirical mean level. In addition, the standard deviation of the VRP is also matched well by the latter two models. However, the Drechsler and Yaron (2008) model has only one state variable. As evident
from our earlier analysis, it is very difficult for this model to explain the volatility predictability cross-sectionally. In addition, studies in the volatility literature (see, e.g., Christoffersen, Jacobs, Ornthalanalai and Wang, 2008, and Lu and Zhu, 2009) show that the two-component model is generally preferred in explaining the the large negative variance risk premium and variance term structure. Hence, our extension of the BY and BKY models seems to offer a promising route for future applications and further extension of the long-run risks models.

Finally, it is of interest to examine the predictability of VRP. Empirically, Drechsler and Yaron (2008) compute an AR(1) coefficient of 0.54, which is largely consistent with Bollerslev and Zhou (2007). Theoretically, the VRP should be predictable due to predictability in volatility. The model implied AR(1) coefficients (see Appendix A.6 for the computation details), reported in the last row of Table V, show that our new model matches remarkably well with the data, while the BY and BKY models are too optimistic on the predictability of VRP. These results may serve as additional evidence for supporting the new model of this paper.

IV. Future Research

The long-run risks models have been developed for the equity market. In comparison with leading models, such as Campbell and Cochrane (1999, 2002) and Barro (2009), that explain the equity risk premium, the long-run risks models seem to have come a long way in explaining in addition a number of other stylized facts about the equity market. With the extension of this paper, this class of models appears to meet most, if not all, of the major issues in equity market. However, there are still three major challenges facing the long-run risks models.

First, a common question is that, to an econometrician, what the long-run risks really stand for. Empirically, they are latent variables to be estimated, and it is difficult to link them to known macroeconomic risks. But this, in our view, might be exactly the advantage of the long-run risks models since no known models can perform well by relying on a few commonly measured macroeconomic variables. In other words, in reality, one variable may be the major driving force at one time, such as oil price in the 70s, while another variable at another time, and sometimes a completely new variable may emerge, such as CDS or those causing financial crisis which, by definition, are unanticipated in a rational economy. The long run risks might capture exactly these
variables over time, i.e., those differing factors that have long and lasting effect on the consumption growth. The “event risk” of Barro (2009) and Liu and Loewenstein (2009) might be imagined as variables of this sort, or the long-run risks with high volatility. As a result, no single name of any macroeconomic variable can be given to the ever present long-run risks.

Second, it is debatable whether there exist long-run persistent fluctuations in consumption and dividend growth rates (see, e.g., the BY, Sargent, 2007, and Beeler and Campbell, 2009). Statistically, given the data size, as far as we know, there is no strong evidence on either side of the debate. Our view is more pragmatic. Without assuming the persistent fluctuations, there have not yet been models that can explain as many stylized facts and as well as the BY does, and hence it will be of great interest and urgency to develop such models to beat the BY. On the other hand, assuming the persistent fluctuations, there is a workable model, which should not be ignored or abandoned simply because of its controversial assumptions. In fact, by studying under either extreme assumptions, it only strengthens our understanding about the driving forces in each type of models. The truth may be somewhere in between, and the models may be patched together in the future.

Thirdly, and perhaps of more practical importance, is that the long-run risks models fail miserably in explaining the fixed income markets. For typically calibrated models, Beeler and Campbell (2009) point out the price of a real consol is unrealistically high and approaches infinity. Our extension of the BY model here of is no exception to this problem either. The reason is that the consumption can have too much drop under some extremely large shocks, so that the real rate can approach zero or even negative to clear the market. As a result, a console that guarantees a unit consumption payoff forever must have a too high price. In addition, Beeler and Campbell (2009) also point out that the risk premia on long-term real bonds relative to short-term ones can have a negative real term premium if consumption growth follows a persistent process. These insights seem to suggest that imposing certain restrictions in the domain, adding a mean-reverting component or allowing some reflecting boundaries can further improve our model here. Bansal and Shaliastovich (2008) provide a novel approach for extension. Moreover, the classic general equilibrium of Cox, Ingersoll, and Ross (1985) appears to offer yet another route via introducing endogenously state variables to explain the fixed income market. Overall, while there remain many challenges, they seem important topics for future research.
V. Conclusion

One of the fundamental problems in asset pricing is to explain various stylized facts about the equity market, for which the rational and general equilibrium long-run risks model of Bansal and Yaron (2004) seems to have come a long way toward this goal. Hence, it is not surprising that there are subsequently a number of important studies along this line of research, including Bansal, Khatchatrian, and Yaron (2005), Bansal, Kiku, and Yaron (2007a, b), Bansal, Dittmar, and Kiku (2008), Drechsler and Yaron (2008), Hansen, Heaton, and Li (2008), Pakos (2008), Avramov and Hore (2009), and Beeler and Campbell (2009). However, in the equity market, there are still three major problems confronting the model. It has difficulty in explaining the predictability of consumption, and the varying degrees of volatility predictability of excess stock returns, consumption and dividends. Moreover, the model fails completely in explaining the large negative market variance risk premium. While Bansal, Kiku, and Yaron (2007b) and Drechsler and Yaron (2008) address one of the problems each, there have been no studies that can resolve all the three problems simultaneously.

This paper is the first to propose such an extension of the Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2007b) models, that solves all three problems simultaneously, while retaining many of the good properties of the original models in explaining well other stylized facts. In our new model, there are two components of volatility: the long-run and short-run volatilities. Both components enter into the dynamics of both the consumption and dividend growth rates. The implied stock volatility process is consistent with findings in the volatility literature, and hence the model can justify the large negative variance risk premium. The flexibility permitted by the two-factor volatility model also allows for explaining the predictability problems. Looking forward, due to wide applications of the long-run risks model of Bansal and Yaron (2004), such as in equity and currency markets, it will be of interest to see how conclusions of these applications might be altered in light of the proposed new model. Additionally, as pointed out by Beeler and Campbell (2009), it is an open challenge to extend the long-run risks models to the fixed income markets. All of these issues are important and exciting topics of future research.
Appendix A

A.1 Derivation for the $A_i$’s

First, we rewrite $f$ as

$$f(C, J) = \frac{\beta}{1 - \frac{1}{\psi}} (1 - \gamma) J [G - 1],$$

where

$$G = \left( \frac{C}{(1 - \gamma) J^{\frac{1}{1 - \psi}}} \right)^{1 - \frac{1}{\psi}}. \quad (A1)$$

Then, taking derivative with respect to $J$ and $C$, we have

$$f_J = (\theta - 1) \beta G - \beta \theta \quad (A2)$$

and

$$f_C = \beta \frac{G}{C} (1 - \gamma) J. \quad (A3)$$

Conjecture a solution for $J$ of the following form,

$$J(W_t, X_t, V_{1t}, V_{2t}) = \exp(A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) W_t^{1 - \gamma}, \quad (A4)$$

and using the standard envelope condition $f_C = J_W$, we have

$$C = J_W^{-\psi} [(1 - \gamma) J]^{\frac{1 - \psi}{1 - \gamma}} \beta^{\psi}. \quad (A5)$$

Substituting (A3) and (A4) into (A5), we obtain

$$\frac{C}{W} = \beta^{\psi} \exp \left[ (A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) \frac{1 - \psi}{1 - \gamma} \right]. \quad (A6)$$

Further substitution of (A6) and (A4) into (A1) implies that

$$\beta G = \frac{C_t}{W_t}. \quad (A7)$$

Applying the log-linear approximation, we obtain

$$\beta G = \frac{C_t}{W_t} \approx g_1 - g_1 \log g_1 + g_1 \log(\beta G). \quad (A7)$$
This implies that
\[ f = \theta J(\theta G - \beta) \approx \theta J \left[ g_1 \frac{1 - \psi}{1 - \gamma} (A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) + \xi \right], \quad \text{(A8)} \]

where \( \theta = 1 - \gamma/(1 - \frac{1}{\psi}) \) and \( \xi = g_1 - g_1 \log g_1 + g_1 \psi \log \beta - \beta \). Substituting (A8) to the Bellman equation, (7), and collecting the terms containing \( X_t, V_{1t} \) and \( V_{2t} \), we have
\[
\theta g_1 \frac{1 - \psi}{1 - \gamma} A_0 + \theta \xi + (1 - \gamma) \mu + \kappa_1 \bar{V}_1 \psi A_2 + \kappa_2 \bar{V}_2 \psi A_3 = 0
\]

\[
X: \quad \theta g_1 \frac{1 - \psi}{1 - \gamma} A_1 + (1 - \gamma) - \alpha \psi A_1 = 0
\]

\[
V_1: \quad \theta g_1 \frac{1 - \psi}{1 - \gamma} A_2 - \frac{1}{2} \gamma(1 - \gamma) \varphi \delta A_1^2 + \frac{1}{2} \dot{\varphi}^2 \delta^2 A_1^2 - \kappa_1 \psi A_2 + \frac{1}{2} \sigma^2 \psi A_2^2 = 0
\]

\[
V_2: \quad \theta g_1 \frac{1 - \psi}{1 - \gamma} A_3 - \frac{1}{2} \gamma(1 - \gamma)(1 - \delta^2) + \frac{1}{2} \dot{\varphi}^2 (1 - \delta^2) \psi^2 A_1^2 - \kappa_2 \psi A_3 + \frac{1}{2} \sigma^2 \psi A_3^2 = 0.
\]

Solving the above algebraic equations, we obtain equation (11) for A’s. Q.E.D.

**A.2 Derivation for the \( A_{im} \)’s**

Let
\[
\frac{D_i}{P_i} = \exp\{(A_{0m} + A_{1m} X_t + A_{2m} V_{1t} + A_{3m} V_{2t})\}.
\]

(A9)

A key step in the derivations is to use the following pricing relation
\[
E_t \left( \frac{dP_i}{P_i} \right) + \frac{D_i}{P_i} dt = r_f dt - E_t \left[ \frac{d\pi_t}{\pi_t} \frac{dP_i}{P_i} \right].
\]

(A10)

With similar loglinear approximation as equation (A7), we can approximate the ratio as
\[
\frac{D_i}{P_i} \approx g_{0m} + g_{1m} \log \frac{D_i}{P_i} = g_{0m} + g_{1m}((A_{0m} + A_{1m} X_t + A_{2m} V_{1t} + A_{3m} V_{2t})),
\]

(A11)

where
\[ g_{0m} = g_{1m} - g_{1m} \log g_{1m}. \]

Applying Ito’s lemma to (A9), we have
\[
\frac{dP_i}{P_i} = \frac{dD_i}{D_i} - (A_{1m} dX_t + A_{2m} dV_{1t} + A_{3m} dV_{2t}) + \frac{1}{2} A_{1m}^2 (dX_t)^2 + \frac{1}{2} A_{2m}^2 (dV_{1t})^2 + \frac{1}{2} A_{3m}^2 (dV_{2t})^2.
\]

Hence,
\[
E_t(\frac{dP_i}{P_i})/dt = \mu_d + \varphi X_t + \alpha A_{1m} X_t - \kappa_1 A_{2m}(\bar{V}_1 - V_{1t}) - \kappa_2 A_{3m}(\bar{V}_2 - V_{2t})
\]
\[
+ \frac{1}{2} A_{1m}^2 \varphi^2 \delta_x [V_{1t} \delta_x^2 + V_{2t}(1 - \delta_x^2)] + \frac{1}{2} A_{2m}^2 \sigma^2 \psi V_{1t} + \frac{1}{2} A_{3m}^2 \sigma^2 \psi V_{2t}.
\]

(A12)
The risk premium term in equation (A10) can thus be written as

\[
-E_t \left[ \frac{d\pi_t}{\pi_t} \frac{dP_t}{P_t} \right] / dt = \lambda_1 \sigma_{dc} \sqrt{V_{1t}\delta_x^2 + V_{2t}(1 - \delta_x^2)} - A_{1m}\lambda_2 \varphi_x \sqrt{V_{1t}\delta_x^2 + V_{2t}(1 - \delta_x^2)} \\
- A_{2m}\lambda_3 \sigma_1 \sqrt{V_{1t}} - A_{3m}\lambda_4 \sigma_2 \sqrt{V_{2t}},
\]

(A13)

where \(\lambda_1, \lambda_2, \lambda_3\) and \(\lambda_4\) are market prices of risk as defined in equation (20), and \(d\pi_t/\pi_t\) is the pricing kernel as defined in (17).

Now, substituting (A11), (A12), (A13), and risk-free rate as defined in (19) into equation (A10), and then collecting terms containing \(X_t\), we obtain

\[A_{1m} = -\frac{\varphi - 1}{g_{1m} + \alpha}.
\]

Collecting terms containing \(V_{1t}\) and \(V_{2t}\), we obtain an equation for \(A_{2m}\),

\[a_{2m}A_{2m}^2 + b_{2m}A_{2m} + c_{2m} = 0
\]

with

\[a_{2m} = \frac{1}{2}\sigma_1^2, \quad b_{2m} = g_{1m} + \kappa_1 - \frac{1 - \gamma \psi}{1 - \gamma} \sigma_2^2, \quad c_{2m} = (\frac{1}{2}A_{1m}^2 - \frac{1 - \gamma \psi}{1 - \gamma} A_{1m})(\varphi_x \delta_x^2)^2 + r_2.
\]

Solving it, we have

\[A_{2m} = \frac{-b_{2m} \pm \sqrt{b_{2m}^2 - 4a_{2m}c_{2m}}}{2a_{2m}}.
\]

We choose the root that goes to zero when \(\sigma_1\) goes to zero. This is because when \(\sigma_1\), or \(a_{3m}\) goes to zero, the price sensitivity to \(V_1\) should be zero.

Similarly, we obtain an equation for \(A_{3m}\),

\[a_{3m}A_{3m}^2 + b_{3m}A_{3m} + c_{3m} = 0
\]

with

\[a_{3m} = \frac{1}{2}\sigma_2^2, \quad b_{3m} = g_{1m} + \kappa_2 - \frac{1 - \gamma \psi}{1 - \gamma} A_{3}\sigma_2^2, \quad c_{3m} = (\frac{1}{2}A_{1m}^2 - \frac{1 - \gamma \psi}{1 - \gamma} A_{1m})\varphi_x^2(1 - \delta_x^2)^2 + r_3.
\]

The solution is

\[A_{3m} = \frac{-b_{3m} \pm \sqrt{b_{3m}^2 - 4a_{3m}c_{3m}}}{2a_{3m}},
\]

where we choose the root in a similar fashion as for \(A_{2m}\).
Finally, collecting the constants, we obtain

\[ \mu_d - \kappa_1 A_{2m} \bar{V}_1 - \kappa_2 A_{3m} \bar{V}_2 + g_{0m} + g_{1m} A_{0m} + r_0 = 0. \]

Then

\[ A_{0m} = -\frac{1}{g_{1m}} \left[ \mu_d - \kappa_1 A_{2m} \bar{V}_1 - \kappa_2 A_{3m} \bar{V}_2 + g_{1m} - g_{1m} \log g_{1m} + r_0 \right], \]

which is the last we need for the log-linear coefficients. Q.E.D.

A.3 Solutions to \( g_1 \) and \( g_{1m} \)

Note that the derived solutions depend on the approximation constant \( g_1 \), but we can solve it endogenously. This is because, given the model parameters, we can compute the unconditional mean of consumption-wealth ratio as a function of the parameters,

\[
g_1 = E \left( \frac{C}{W} \right) = \beta^\psi \exp\{A_0a\} \exp\left\{ \frac{1}{\alpha} \frac{A_{2a}^2 \bar{V}_1 \delta_x^2 + \bar{V}_2 (1 - \delta_x^2)}{2 \kappa_1 / \sigma_1^2} \right\} \cdot \exp\left\{ -\frac{2 \kappa_1 \bar{V}_1}{\sigma_1^2} \log(1 - \frac{A_{2a}}{2 \kappa_1 / \sigma_1^2}) \right\} \cdot \exp\left\{ -\frac{2 \kappa_2 \bar{V}_2}{\sigma_2^2} \log(1 - \frac{A_{3a}}{2 \kappa_2 / \sigma_2^2}) \right\}. \tag{A14}
\]

Now note that the \( A_{ia} \)'s on the right hand side are also a function of \( g_1 \). Plugging in, we obtain a nonlinear function in terms of \( g_1 \) alone, and hence \( g_1 \) can be solved in terms of the fundamental parameters of the model, and can be solve numerically with many available algorithms.

Similarly, we can solve \( g_{1m} \) endogenously based on dividend-price ratio is

\[
g_{1m} = E \frac{D}{P} = \exp\{A_{0m}\} \exp\left\{ \frac{1}{\alpha} \frac{A_{2m}^2 \bar{V}_1 \delta_x^2 + \bar{V}_2 (1 - \delta_x^2)}{2 \kappa_1 / \sigma_1^2} \right\} \cdot \exp\left\{ -\frac{2 \kappa_1 \bar{V}_1}{\sigma_1^2} \log(1 - \frac{A_{2m}}{2 \kappa_1 / \sigma_1^2}) \right\} \cdot \exp\left\{ -\frac{2 \kappa_2 \bar{V}_2}{\sigma_2^2} \log(1 - \frac{A_{3m}}{2 \kappa_2 / \sigma_2^2}) \right\}. \tag{A15}
\]

This is easy to solve numerically. Q.E.D.
A.4 Predictability of Variables

First, we want to show (29). Note that the log price process can be written as:

\[ d \log P_t = \left[ \mu_d - (A_{2m} \kappa_1 \bar{V}_1 + A_{3m} \kappa_2 \bar{V}_2) + (\varphi + \alpha A_{1m}) X_t + (A_{2m} \kappa_1 - \frac{1}{2} \delta_c^2) V_{1t} \right] dt + \sqrt{\delta_c^2 V_{1t} + (1 - \delta_c^2) V_{2t}} dZ_t, \]  

(A16)

the log consumption process can be written as:

\[ d \log C_t = \left[ \mu + X_t - \frac{1}{2} \delta_c^2 V_{1t} - \frac{1}{2} (1 - \delta_c^2) V_{2t} \right] dt + \sqrt{\delta_c^2 V_{1t} + (1 - \delta_c^2) V_{2t}} dZ_{1t}, \]  

(A17)

and the log dividend process can be written as:

\[ d \log D_t = \left[ \mu_d + \varphi X_t - \frac{1}{2} (\varphi^2 \delta_d^2 + \sigma^2_{dc} \delta_c^2) V_{1t} - \frac{1}{2} (\varphi^2 (1 - \delta_d^2) \delta_c^2) V_{2t} \right] dt + \sqrt{\delta_d^2 V_{1t} + \sigma^2_{dc} V_{2t}} dB_t + \sigma_{dc} \sqrt{V_{1t} \delta_d^2 + V_{2t}(1 - \delta_d^2)} dZ_{1t}, \]  

(A18)

Combining these with the riskfree rate process and their suitable differences, we obtain (29) as well as \( p_t - d_t \). Then, the covariance and variance evaluations are mechanical, though tedious, and the results are given as (31) and (32). Q.E.D.

A.5 Predictability of Volatilities

First, we prove equation (38). To do so, we apply the following approximation:

\[ \frac{1}{\tau} \int_0^\tau \exp(x_s) ds \approx \exp\left( \frac{1}{\tau} \int_0^\tau x_s ds \right) \]  

(A20)

for any process \( x_s \). This is equivalent to an approximation of arithmetic mean by geometric mean. It is a good approximation when the variation of \( x_t \) is small in magnitude both across time and probability space. This is true for our variance processes because the magnitude of the variance is generally to the order of \( 10^{-3} \sim 10^{-4} \), and the variation of \( \ln V_t \) is within 1. Applying the above approximation to \( \ln V_t \), we have

\[ \frac{1}{\tau} \int_0^\tau \sqrt{V_t} dt = \frac{1}{\tau} \int_0^\tau \exp\left( \frac{1}{2} \ln V_t \right) dt \approx \exp\left( \frac{1}{2\tau} \int_0^{\tau} \ln V_t dt \right). \]  

(A21)
Hence,

\[
\ln \frac{1}{\tau} \int_0^\tau \sqrt{V_t} \, dt \approx \frac{1}{2\tau} \int_0^\tau \ln V_t \, dt
\]

\[
= \frac{1}{2\tau} \left[ \int_0^\tau \ln \bar{V} + \int_0^\tau \ln \left(1 + \frac{V_t - \bar{V}}{V}\right) \, dt \right]
\]

\[
\approx \frac{1}{2} \ln \bar{V} + \frac{1}{2\tau} \int_0^\tau \frac{V_t - \bar{V}}{V} \, dt
\]

\[
= \text{Const} + \frac{1}{2\tau V} \int_0^\tau V_t \, dt,
\]

which is equation (38).

Because of the approximation above, we can express the volatilities as an integral of \( b_1 V_1 + b_2 V_2 \) over \( (t, t + \tau) \). Plugging in the terms into the definition of the covariance, we then obtain (40). Q.E.D.

### A.6 Derivation of the AR(1) Coefficient

Consider a stochastic process of the form

\[
b_1 V_{1t} + b_2 V_{2t},
\]

where \( b_1 \) and \( b_2 \) are constants. Due to independence between \( V_1 \) and \( V_2 \), the unconditional autocovariance can be evaluated as

\[
b_1^2 \sigma_1^2 \frac{\bar{V}_1}{2\kappa_1} \exp(-\kappa_1 \tau) + b_2^2 \sigma_2^2 \frac{\bar{V}_2}{2\kappa_2} \exp(-\kappa_2 \tau)
\]

and the unconditional variance can be evaluated as

\[
b_1^2 \sigma_1^2 \frac{\bar{V}_1}{2\kappa_1} + b_2^2 \sigma_2^2 \frac{\bar{V}_2}{2\kappa_2}.
\]

Hence, the AR(1) coefficient can be computed easily based on above. Q.E.D.
References

Avramov, Doron, and Satadru Hore, 2009, Momentum, information uncertainty, and leverage -an explanation based on recursive preferences, Working paper, University of Maryland.


Table I: Long Run Risks Parameters

The table reports the parameters for the three calibrated long-run risks models: the Bansal and Yaron (BY, 2004), and Bansal, Kiku, and Yaron (BKY, 2007), and the new model. In Panel A, \( \gamma \) is the risk aversion parameter, \( \psi \) the EIS parameter, \( \beta \) the discount rate. Other panels provide parameters governing the consumption, dividend and volatility dynamics:

\[
\begin{align*}
\frac{dC_t}{C_t} &= (\mu + X_t)dt + \sqrt{V_{1t}\delta_c^2 + V_{2t}(1 - \delta_c^2)}dZ_{1t} \\
\frac{dX_t}{C_t} &= -\alpha X_t dt + \varphi_x \sqrt{V_{1t}\delta_x^2 + V_{2t}(1 - \delta_x^2)}dZ_{2t} \\
\frac{dD_t}{D_t} &= (\mu_d + \varphi X_t)dt + \varphi_d \sqrt{V_{1t}\delta_d^2 + V_{2t}(1 - \delta_d^2)}dB_t + \sigma_{dc} \sqrt{V_{1t}\delta_c^2 + V_{2t}(1 - \delta_c^2)}dZ_{1t} \\
&+ \sigma_{dx} \sqrt{V_{1t}\delta_x^2 + V_{2t}(1 - \delta_x^2)}dZ_{2t} + \sigma_{dv} \sqrt{V_{1t}}dw_{1t} \\
\frac{dV_{1t}}{V_{1t}} &= \kappa_1(V_1 - V_{1t})dt + \sigma_1 \sqrt{V_{1t}}dw_{1t} \\
\frac{dV_{2t}}{V_{2t}} &= \kappa_2(V_2 - V_{2t})dt + \sigma_2 \sqrt{V_{2t}}dw_{2t}, \quad \kappa_1 < \kappa_2,
\end{align*}
\]

where the parameter values are annualized whenever applicable.

<table>
<thead>
<tr>
<th>Preference Parameters</th>
<th>( \gamma )</th>
<th>( \psi )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BY</td>
<td>7.5 – 10</td>
<td>1.5</td>
<td>2.4%</td>
</tr>
<tr>
<td>BKY</td>
<td>10</td>
<td>1.5</td>
<td>3%</td>
</tr>
<tr>
<td>New</td>
<td>6</td>
<td>1.5</td>
<td>2.4%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Consumption Growth Dynamics</th>
<th>( \mu )</th>
<th>( \alpha )</th>
<th>( \varphi_x )</th>
<th>( \delta_c )</th>
<th>( \delta_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BY</td>
<td>0.018</td>
<td>0.256</td>
<td>0.528</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>BKY</td>
<td>0.018</td>
<td>0.3</td>
<td>0.456</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>New</td>
<td>0.018</td>
<td>0.256</td>
<td>0.58</td>
<td>0.8</td>
<td>0.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dividend Growth Dynamics</th>
<th>( \mu_d )</th>
<th>( \phi )</th>
<th>( \phi_d )</th>
<th>( \delta_d )</th>
<th>( \sigma_{dc} )</th>
<th>( \sigma_{dx} )</th>
<th>( \sigma_{dv} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BY</td>
<td>0.018</td>
<td>3</td>
<td>4.5</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BKY</td>
<td>0.018</td>
<td>2.5</td>
<td>5.96</td>
<td>1</td>
<td>2.6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>New</td>
<td>0.03</td>
<td>3.5</td>
<td>3.5</td>
<td>0</td>
<td>1</td>
<td>-4.5</td>
<td>1.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volatility Parameters</th>
<th>Factor 1</th>
<th>Factor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_1 )</td>
<td>( \sigma_1 )</td>
<td>( \kappa_1 )</td>
</tr>
<tr>
<td>BY</td>
<td>0.027^2</td>
<td>0.0035</td>
</tr>
<tr>
<td>BKY</td>
<td>0.025^2</td>
<td>0.0027</td>
</tr>
<tr>
<td>New</td>
<td>0.022^2</td>
<td>0.004</td>
</tr>
</tbody>
</table>
Table II: Long Run Risks Moments

The table reports moments of various variables of the Long-run risks model. The second column provides those computed based on monthly data from February, 1947 to March, 2007; the third column provides those based on the BY model; and the fourth those based on the BKY model. These results in the three columns are from Beeler and Campbell (2009). The last column provides the moments based on our new model.

<table>
<thead>
<tr>
<th>Moment</th>
<th>Data (1947.2-2007.3)</th>
<th>BY</th>
<th>BKY</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(\Delta c)$</td>
<td>1.79</td>
<td>1.95</td>
<td>1.82</td>
<td>1.76</td>
</tr>
<tr>
<td>$\sigma(\Delta c)$</td>
<td>2.16</td>
<td>2.92</td>
<td>2.96</td>
<td>2.96</td>
</tr>
<tr>
<td>AC1($\Delta c$)</td>
<td>0.44</td>
<td>0.51</td>
<td>0.44</td>
<td>0.33</td>
</tr>
<tr>
<td>$E(\Delta d)$</td>
<td>1.02</td>
<td>1.66</td>
<td>1.85</td>
<td>1.21</td>
</tr>
<tr>
<td>$\sigma(\Delta d)$</td>
<td>10.69</td>
<td>11.57</td>
<td>16.42</td>
<td>18.93</td>
</tr>
<tr>
<td>AC1($\Delta d$)</td>
<td>0.14</td>
<td>0.40</td>
<td>0.29</td>
<td>0.11</td>
</tr>
<tr>
<td>$E(r_e)$</td>
<td>6.2</td>
<td>6.62</td>
<td>6.58</td>
<td>3.58</td>
</tr>
<tr>
<td>$\sigma(r_e)$</td>
<td>18.34</td>
<td>16.88</td>
<td>21.35</td>
<td>18.33</td>
</tr>
<tr>
<td>AC1($r_e$)</td>
<td>0.04</td>
<td>0.03</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>$E(r_f)$</td>
<td>0.99</td>
<td>2.56</td>
<td>0.99</td>
<td>2.63</td>
</tr>
<tr>
<td>$\sigma(r_f)$</td>
<td>4.28</td>
<td>1.30</td>
<td>1.28</td>
<td>4.14</td>
</tr>
<tr>
<td>AC1($r_f$)</td>
<td>0.59</td>
<td>1.28</td>
<td>0.86</td>
<td>0.11</td>
</tr>
<tr>
<td>$E(p-d)$</td>
<td>3.31</td>
<td>3.00</td>
<td>3.04</td>
<td>2.75</td>
</tr>
<tr>
<td>$\sigma(p-d)$</td>
<td>0.46</td>
<td>0.16</td>
<td>0.26</td>
<td>0.42</td>
</tr>
<tr>
<td>AC1($p-d$)</td>
<td>0.88</td>
<td>0.77</td>
<td>0.95</td>
<td>0.94</td>
</tr>
</tbody>
</table>
Table III: Predictability of Excess Returns, Consumption and Dividends

The table reports the slope coefficients from predictive regressions of excess returns, consumption growth and dividend growth on log price-dividend ratios based on the data, the BY, the BKY and our new models. Except the last column, the results are from Beeler and Campbell (2009).

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>BY</th>
<th>BKY</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Excess return</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1Y</td>
<td>-0.059</td>
<td>-0.007</td>
<td>-0.078</td>
<td>-0.074</td>
</tr>
<tr>
<td>3Y</td>
<td>-0.229</td>
<td>-0.026</td>
<td>-0.226</td>
<td>-0.214</td>
</tr>
<tr>
<td>5Y</td>
<td>-0.421</td>
<td>-0.039</td>
<td>-0.368</td>
<td>-0.351</td>
</tr>
<tr>
<td><strong>Consumption growth</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1Y</td>
<td>0.012</td>
<td>0.114</td>
<td>0.022</td>
<td>0.022</td>
</tr>
<tr>
<td>3Y</td>
<td>0.010</td>
<td>0.286</td>
<td>0.052</td>
<td>0.051</td>
</tr>
<tr>
<td>5Y</td>
<td>-0.001</td>
<td>0.388</td>
<td>0.069</td>
<td>0.069</td>
</tr>
<tr>
<td><strong>Dividend</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1Y</td>
<td>0.064</td>
<td>0.343</td>
<td>0.054</td>
<td>0.088</td>
</tr>
<tr>
<td>3Y</td>
<td>0.076</td>
<td>0.860</td>
<td>0.133</td>
<td>0.213</td>
</tr>
<tr>
<td>5Y</td>
<td>0.051</td>
<td>1.171</td>
<td>0.176</td>
<td>0.297</td>
</tr>
</tbody>
</table>
Table IV: Predictability of Volatility: Excess Returns, Consumption and Dividends
The table reports the slope coefficients from predictive regressions of volatilities (of excess returns, consumption growth and dividend growth) on log price-dividend ratios based on the data, the BY, the BKY and our new models. Except the last column, the results are from Beeler and Campbell (2009).

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>BY</th>
<th>BKY</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excess return volatility</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1Y</td>
<td>-0.081</td>
<td>-0.123</td>
<td>-1.315</td>
<td>-0.133</td>
</tr>
<tr>
<td>3Y</td>
<td>-0.059</td>
<td>-0.115</td>
<td>-1.268</td>
<td>-0.064</td>
</tr>
<tr>
<td>5Y</td>
<td>-0.017</td>
<td>-0.113</td>
<td>-1.336</td>
<td>-0.050</td>
</tr>
<tr>
<td>Consumption volatility</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1Y</td>
<td>-0.481</td>
<td>-0.128</td>
<td>-1.420</td>
<td>-0.496</td>
</tr>
<tr>
<td>3Y</td>
<td>-0.491</td>
<td>-0.122</td>
<td>-1.382</td>
<td>-0.446</td>
</tr>
<tr>
<td>5Y</td>
<td>-0.564</td>
<td>-0.113</td>
<td>-1.336</td>
<td>-0.431</td>
</tr>
<tr>
<td>Dividend volatility</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1Y</td>
<td>-0.530</td>
<td>-0.146</td>
<td>-1.483</td>
<td>-0.476</td>
</tr>
<tr>
<td>3Y</td>
<td>-0.478</td>
<td>-0.144</td>
<td>-1.431</td>
<td>-0.425</td>
</tr>
<tr>
<td>5Y</td>
<td>-0.496</td>
<td>-0.123</td>
<td>-1.384</td>
<td>-0.410</td>
</tr>
</tbody>
</table>

Table V: Variance Risk Premium
The table reports variance risk premiums from data, the Drechsler and Yaron (2008), the BY, the BKY and our new models. The results for the first two are from Drechsler and Yaron (2008). All the values are monthly and multiplied by 10000.

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>DY</th>
<th>BY</th>
<th>BKY</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>VRP</td>
<td>-12.67</td>
<td>-7.57</td>
<td>-0.005</td>
<td>-0.010</td>
<td>-6.04</td>
</tr>
<tr>
<td>std</td>
<td>14.38</td>
<td>10.65</td>
<td>0.000</td>
<td>0.000</td>
<td>7.31</td>
</tr>
<tr>
<td>AR1</td>
<td>0.54</td>
<td>N/A</td>
<td>0.99</td>
<td>0.99</td>
<td>0.58</td>
</tr>
</tbody>
</table>